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A NOTE ON A CLASS OF BANACH ALGEBRA-VALUED POLYNOMIALS

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Let F be a Banach algebra. We give a necessary and sufficient condition for F to be finite dimensional, in terms of finite type n-homogeneous F-valued polynomials.

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1. Introduction and results. Let E and F be complex Banach spaces. We denote by $L(^nE,F)$ the Banach space of all continuous n-linear mappings A from E^n into F endowed with the norm $\|A\| = \sup\{\|A(x_1,\dots,x_n)\| : \|x_j\| \le 1,\ j = 1,\dots,n\}$. A mapping P from E into F is called a continuous n-homogeneous polynomial if $P(x) = A(x,\dots,x)$ (for all $x \in E$) for some $A \in L(^nE,F)$. We denote by $P(^nE,F)$ the Banach space of all continuous n-homogeneous polynomials P from E into F endowed with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \le 1\}$. Also a mapping P from E into F is called a finite type n-homogeneous polynomial if $P(x) = f_1(x)^n b_1 + \cdots + f_k(x)^n b_k$ (for all $x \in E$), where $f_1,\dots,f_k \in E^*$ and $b_1,\dots,b_k \in F$. We denote by $P_f(^nE,F)$ the space of all finite type n-homogeneous polynomials P from E into F. Then we have $P_f(^nE,F) \subseteq P(^nE,F)$. Indeed, let $P \in P_f(^nE,F)$. Then we write $P(x) = f_1(x)^n b_1 + \cdots + f_k(x)^n b_k$ $(x \in E)$ for some $f_1,\dots,f_k \in E^*$ and $b_1,\dots,b_k \in F$. Set

$$A(x_1,\dots,x_n) = \sum_{i=1}^{k} f_i(x_1)\cdots f_i(x_n) b_i, \quad (x_1,\dots,x_n) \in E^n. \tag{1.1}$$

Then A is a continuous n-linear mapping from E^n into F and $P(x) = A(x,\dots,x)$ $(x \in E)$. Hence $P \in P(^nE,F)$. We are now interested in the case that F is a Banach algebra. Let

$$\mathbf{P}_f(^nE,F) = \{\varphi_1^n + \cdots + \varphi_k^n : \varphi_j \in B(E,F)\ (j = 1,\dots,k),\ k \in \mathbb{N}\}, \tag{1.2}$$

where $\varphi_j^n(x) = (\varphi_j(x))^n$ $(x \in E)$. Then we have $\mathbf{P}_f(^nE,\mathbf{C}) = P_f(^nE,\mathbf{C})$ and $\mathbf{P}_f(^n\mathbf{C},F) \subseteq P_f(^n\mathbf{C},F)$ (see [1, Section 1]). Also, we have $\mathbf{P}_f(^nE,F) \subseteq P(^nE,F)$. Indeed, let $P \in \mathbf{P}_f(^nE,F)$. Then we can write $P = \varphi_1^n + \cdots + \varphi_k^n$ for some $\varphi_1,\dots,\varphi_k \in B(E,F)$. Set $A(x_1,\dots,x_n) = \sum_{i=1}^{k} \varphi_i(x_1)\cdots\varphi_i(x_n)$, $(x_1,\dots,x_n) \in E^n$. Then A is a continuous n-linear mapping from E^n into F and $P(x) = A(x,\dots,x)$ $(x \in E)$. Hence $P \in P(^nE,F)$.

Now, for each $n \in \mathbb{N}$, we say that an algebra F has the r_n-property if, given any $b \in F$, we can find elements $a_1,\dots,a_p \in F$ such that $b = \sum_{i=1}^{p} a_i^n$. We also say that an algebra F has the r-property if F has the r_n-property for each $n \in \mathbb{N}$.

Proposition 1.1 (see [1]). (1) *Every unital complex algebra has the r-property.*

(2) *Let E be a Banach space and F be a Banach algebra. Then $P_f({}^nE,F) \subseteq \mathbf{P}_f({}^nE,F)$ if and only if F has the r_n-property.*

In [1], it is remarked that, given an arbitrary Banach space $(F,+,\|\cdot\|)$, we can always define a product $\circ$ and a norm $\|\cdot\|_*$ on F in order that $(F,+,\circ,\|\cdot\|_*)$ is a unital Banach algebra and $\|\cdot\|_*$ is equivalent to $\|\cdot\|$. By Proposition 1.1 and the above remark, Lourenço-Moraes proved the following proposition.

Proposition 1.2 (see [1]). *Let E be a Banach space. The following are equivalent:*

(a) *E is a finite-dimensional space;*

(b) *$P_f({}^nE,F) = \mathbf{P}_f({}^nE,F)$ for every $n \in \mathbb{N}$ and for every Banach algebra F with the r_n-property;*

(c) *$P_f({}^nE,F) = \mathbf{P}_f({}^nE,F)$ for every $n \in \mathbb{N}$ and for every unital Banach algebra F.*

Remark 1.3. By the proof of Proposition 1.2 (see [1]), we see that each of the following two statements are also equivalent to one of, hence all of, (a), (b), and (c) in Proposition 1.2:

(b′) $P_f({}^1E,F) = \mathbf{P}_f({}^1E,F)$ for every unital Banach algebra F;

(d) $P_f({}^nE,F) = \mathbf{P}_f({}^nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space F.

In this note we show the following result, which is opposite to Proposition 1.2.

Proposition 1.4. *Let F be a Banach algebra. Then the following are equivalent:*

(a) *F is a finite-dimensional space;*

(b) *$\mathbf{P}_f({}^nE,F) \subseteq P_f({}^nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space E;*

(c) *$\mathbf{P}_f({}^1E,F) \subseteq P_f({}^1E,F)$ for every Banach space E.*

In particular, in the unital case, we have the following proposition.

Proposition 1.5. *Let F be a unital Banach algebra. Then the following are equivalent:*

(a) *F is a finite-dimensional space;*

(b) *$\mathbf{P}_f({}^nE,F) = P_f({}^nE,F)$ for every $n \in \mathbb{N}$ and for every Banach space E;*

(c) *$\mathbf{P}_f({}^1E,F) = P_f({}^1E,F)$ for every Banach space E.*

2. Proofs

Lemma 2.1. *Let n be any positive integer and let $x_1,\dots,x_n$ be n-real variables. Then*

$$\prod_{i=1}^{n} x_i = \frac{1}{2^n n!} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1 \cdots \varepsilon_n \left(\sum_{k=1}^{n} \varepsilon_k x_k \right)^n \tag{2.1}$$

holds.

Proof. For each m with $0 \le m \le n$, let

$$P_m(x_1,\dots,x_n) = \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1 \cdots \varepsilon_n \left(\sum_{k=1}^{n} \varepsilon_k x_k \right)^m. \tag{2.2}$$

Then we have $P_m(0,x_2,\dots,x_n) = P_m(x_1,0,\dots,x_n) = \cdots = P_m(x_1,\dots,x_{n-1},0) = 0$. Indeed since

$$\begin{aligned} P_m(x_1,\dots,x_n) = &\sum_{\varepsilon_2,\dots,\varepsilon_n=\pm 1} \varepsilon_2\cdots\varepsilon_n(x_1+\varepsilon_2 x_2+\cdots+\varepsilon_n x_n)^m \\ &- \sum_{\varepsilon_2,\dots,\varepsilon_n=\pm 1} \varepsilon_2\cdots\varepsilon_n(-x_1+\varepsilon_2 x_2+\cdots+\varepsilon_n x_n)^m, \end{aligned} \tag{2.3}$$

it follows that $P_m(0,x_2,\dots,x_n) = 0$. Similarly,

$$P_m(x_1,0,\dots,x_n) = \cdots = P_m(x_1,\dots,x_{n-1},0) = 0. \tag{2.4}$$

Therefore, we have

$$P_m(x_1,\dots,x_n) = 0, \tag{2.5}$$

for each $m = 0,1,2,\dots,n-1$ and

$$P_n(x_1,\dots,x_n) = K_n \prod_{i=1}^{n} x_i, \tag{2.6}$$

for some constant K_n, because $P_m(x_1,\dots,x_n)$ is m-homogeneous for $x_1,\dots,x_n$. Hence we only show that $K_n = 2^n n!$. Note that

$$K_n = P_n(1,\dots,1) = \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n \left(\sum_{k=1}^{n} \varepsilon_k\right)^n. \tag{2.7}$$

Then $K_1 = 2$. Now, for each m with $0 \le m \le n$, let $\alpha_m = \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n(\sum_{k=1}^{n}\varepsilon_k)^m$. Then by (2.5) and (2.6), we have $\alpha_0 = \alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n = K_n$. Hence,

$$\begin{aligned} K_{n+1} &= \sum_{\varepsilon_1,\dots,\varepsilon_{n+1}=\pm 1} \varepsilon_1\cdots\varepsilon_{n+1}\left(\sum_{k=1}^{n+1}\varepsilon_k\right)^{n+1} \\ &= \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n\left(\sum_{k=1}^{n}\varepsilon_k+1\right)^{n+1} - \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n\left(\sum_{k=1}^{n}\varepsilon_k-1\right)^{n+1} \\ &= \sum_{m=0}^{n+1}\binom{n+1}{m} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n\left(\sum_{k=1}^{n}\varepsilon_k\right)^m \\ &\quad - \sum_{m=0}^{n+1}\binom{n+1}{m} \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n(-1)^{n+1-m}\left(\sum_{k=1}^{n}\varepsilon_k\right)^m \\ &= \sum_{m=0}^{n+1}\binom{n+1}{m}\left(1-(-1)^{n+1-m}\right) \sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1} \varepsilon_1\cdots\varepsilon_n\left(\sum_{k=1}^{n}\varepsilon_k\right)^m \\ &= \sum_{m=0}^{n}\binom{n+1}{m}\left(1-(-1)^{n+1-m}\right)\alpha_m \\ &= \binom{n+1}{n}\left(1-(-1)^{n+1-n}\right)K_n \\ &= 2(n+1)K_n, \end{aligned} \tag{2.8}$$

so that we have $K_n = 2^n n!$ $(n = 1,2,\dots)$ inductively. □

Proof of Proposition 1.4. (a)⇒(b). Let $\{u_1,\dots,u_N\}$ be a basis of F and $g_1,\dots,g_N$ the corresponding coordinate functionals, that is, $g_i(u_j)=\delta_{ij}$ $(i,j=1,\dots,N)$. Let $P\in \mathbf{P}_f({}^nE,F)$. Then we can write $P(x)=\sum_{i=1}^{\ell}(T_i(x))^n$ $(x\in E)$ for some $T_1,\dots,T_\ell\in B(E,F)$. Let

$$f_{ij}(x)=g_j(T_i(x)) \quad (x\in E), \tag{2.9}$$

for each $i=1,\dots,\ell$, $j=1,\dots,N$. Then we have $T_i(x)=\sum_{j=1}^{N} f_{ij}(x)u_j$ $(x\in E,\ i=1,\dots,\ell)$, and hence by Lemma 2.1,

$$\begin{aligned} P(x) &= \sum_{i=1}^{\ell}\left(\sum_{j=1}^{N} f_{ij}(x)u_j\right)^n \\ &= \sum_{i=1}^{\ell}\sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N} f_{ij_1}(x)\cdots f_{ij_n}(x)u_{j_1}\cdots u_{j_n} \\ &= \sum_{i=1}^{\ell}\sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N}\frac{1}{K_n}\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm1}\varepsilon_1\cdots\varepsilon_n\left(\sum_{k=1}^{n}\varepsilon_k f_{ij_k}(x)\right)^n u_{j_1}\cdots u_{j_n} \\ &= \sum_{i=1}^{\ell}\sum_{j_1=1}^{N}\cdots\sum_{j_n=1}^{N}\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm1}\left(f_{i,j_1,\dots,j_n,\varepsilon_1,\dots,\varepsilon_n}(x)\right)^n b_{j_1,\dots,j_n,\varepsilon_1,\dots,\varepsilon_n}, \end{aligned} \tag{2.10}$$

for each $x\in E$, where $f_{i,j_1,\dots,j_n,\varepsilon_1,\dots,\varepsilon_n}=\varepsilon_1 f_{ij_1}+\cdots+\varepsilon_n f_{ij_n}\in E^*$ and $b_{j_1,\dots,j_n,\varepsilon_1,\dots,\varepsilon_n}=(1/K_n)\varepsilon_1\cdots\varepsilon_n u_{j_1}\cdots u_{j_n}\in F$. Therefore we have $P\in P_f({}^nE,F)$.

(b)⇒(c). This is trivial.

(c)⇒(a). Suppose that $\mathbf{P}_f({}^1E,F)\subseteq P_f({}^1E,F)$ for every Banach space E. Note that $P_f({}^1F,F)=\{T\in B(F,F):\dim T(F)<\infty\}$ and $\mathbf{P}_f({}^1F,F)=B(F,F)$. Then by hypothesis, the identity map of F onto itself is finite dimensional and so is F. □

Proof of Proposition 1.5. This follows immediately from Propositions 1.1 and 1.4. □

References

[1] M. L. Lourenço and L. A. Moraes, *A class of polynomials from Banach spaces into Banach algebras*, Publ. Res. Inst. Math. Sci. **37** (2001), no. 4, 521–529.

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