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A MINIMIZATION THEOREM IN QUASI-METRIC SPACES AND ITS APPLICATIONS

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We prove a new minimization theorem in quasi-metric spaces, which improves the results of Takahashi (1993). Further, this theorem is used to generalize Caristi's fixed point theorem and Ekeland's ε-variational principle.

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1. Introduction. Caristi [1] proved a fixed point theorem on complete metric spaces which generalizes the Banach contraction principle. Ekeland [3] also obtained a nonconvex minimization theorem, often called the ε-variational principle, for a proper lower semicontinuous function, bounded from below, on complete metric spaces. Later Takahashi [4] proved the following minimization theorem: let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u,v) \le f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. These theorems are very useful tools in nonlinear analysis, control theory, economic theory, and global analysis.

2. Main results. Throughout this note, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Definition 2.1. A real-valued function Φ defined on a topological space X is said to be lower semicontinuous at x in X if and only if $\{x_\lambda\}$ is a net in X and $\lim x_\lambda = x$ implies $\Phi x \le \liminf \Phi x_\lambda$.

Definition 2.2 [2]**.** A real-valued function Φ defined on a topological space X is said to be weak lower semicontinuous at $x \in X$ if and only if $\{x_\lambda\}$ is a net in X and $\lim x_\lambda = x$ implies $\Phi x \le \limsup \Phi x_\lambda$. A mapping Φ is said to be a weak lower semicontinuous on X if and only if it is weak lower semicontinuous for every $x \in X$.

Definition 2.3. A pair (X,d) of a set X and a mapping d from $X \times X$ into real numbers is said to be a quasi-metric space if and only if

$$\begin{aligned} &d(x,y) \ge 0, \quad d(x,y) = 0 \quad \text{iff } x = y, \\ &d(x,z) \le d(x,y) + d(y,z) \quad \forall x,y,z \in X. \end{aligned} \tag{2.1}$$

Definition 2.4. A sequence $\{x_n\}$ in X is said to be a left k-Cauchy sequence if for each $k \in \mathbb{N}$ there is an N_k such that

$$d(x_n, x_m) < \frac{1}{k} \quad \forall m \geq n \geq N_k. \tag{2.2}$$

A quasi-metric space is left k-sequentially complete if each left k-Cauchy sequence is convergent.

Theorem 2.5. *Let (X,d) be left k-sequentially complete quasi metric space such that for each $x \in X$ the mapping $u \to d(x,u)$ is a lower semicontinuous on X. Let $f : X \to (-\infty, \infty]$ be a proper weak lower semicontinuous function bounded from below such that for any $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ with $v \neq u$ and $f(v)+d(u,v) \leq f(u)$. Then there exists $x_0 \in X$ such that $\inf_{x\in X} f(x) = f(x_0)$.*

Proof. Suppose that $\inf_{x\in X} f(x) < f(y)$ for every $y \in X$. For each $y \in X$, we define $S(y)$ by

$$S(y) = \{z \in X : f(z)+d(y,z) \leq f(y)\}. \tag{2.3}$$

From (2.3) and hypotheses of the theorem we have the following:

$(*)$ For each $y \in X$, there exists $v \in X$ with $v \neq y$ such that $v \in S(y)$, and for each $z \in S(y)$, $S(z) \subseteq S(y)$.

For each $y \in X$, we define $A(y)$ by

$$A(y) = \inf\{f(z) : z \in S(y)\}. \tag{2.4}$$

Choose $u \in X$ with $f(u) < \infty$. Then we choose a sequence $\{u_n\}$ in $S(u)$ as follows: when $u = u_1, u_2, \ldots, u_n$ have been chosen, choose $u_{n+1} \in S(u_n)$ such that

$$f(u_{n+1}) < A(u_n) + \frac{1}{n}. \tag{2.5}$$

Thus, we obtain a sequence $\{u_n\}$ such that

$$d(u_n, u_{n+1}) \leq f(u_n) - f(u_{n+1}), \tag{2.6}$$

$$f(u_{n+1}) - \frac{1}{n} < A(u_n) \leq f(u_{n+1}). \tag{2.7}$$

By (2.6), $\{f(u_n)\}$ is a nonincreasing sequence of reals and so it converges. Therefore, by (2.7) there is some α in $\mathbb{R}$ such that

$$\alpha = \lim_{n\to\infty} A(u_n) = \lim_{n\to\infty} f(u_n) = \inf_{n\in N} f(u_n). \tag{2.8}$$

Let $k \in \mathbb{N}$ be arbitrary. By (2.8) there exists some N_k such that $f(u_n) < \alpha + 1/k$ for all $n \geq N_k$. Thus, by monotony of $\{f(u_n)\}$, for $m \geq n \geq N_k$, we have

$$\alpha \leq f(u_m) \leq f(u_n) < \alpha + \frac{1}{k}, \tag{2.9}$$

and hence

$$f(u_n) - f(u_m) < \frac{1}{k} \quad \forall m > n \geq N_k. \tag{2.10}$$

From the triangle inequality, (2.6) and (2.10), we get

$$\begin{aligned} d(u_n,u_m) &\le \sum_{i=n}^{m-1} d(u_i,u_{i+1}) \le \sum_{i=n}^{m-1} [f(u_i)-f(u_{i+1})] \\ &\le f(u_n)-f(u_m) < \frac{1}{k} \end{aligned} \tag{2.11}$$

for all $m > n \ge N_k$.

Therefore, $\{u_n\}$ is a left k-Cauchy sequence in X. By completeness, there exists $z \in X$ such that $u_n \to z$. Since f is a weak lower semicontinuous; by (2.8), we have

$$f(z) \le \limsup_{n\to\infty} f(u_n) = \alpha. \tag{2.12}$$

From (2.11), we obtain

$$f(u_m) \le f(u_n) - d(u_n,u_m). \tag{2.13}$$

Since f is a weak lower semicontinuous on X and $u \to d(x,u)$ on X is a lower semicontinuous, we have

$$\begin{aligned} f(z) &\le \limsup_{m\to\infty} f(u_m) \le f(u_n) + \limsup_{m\to\infty} [-d(u_n,u_m)] \\ &= f(u_n) - \liminf_{m\to\infty} d(u_n,u_m) = f(u_n) - d(u_n,z). \end{aligned} \tag{2.14}$$

Hence

$$d(u_n,z) \le f(u_n) - f(z). \tag{2.15}$$

From (2.3) and (2.15), we obtain that $z \in S(u_n)$ for every $n \in \mathbb{N}$ and hence

$$A(u_n) \le f(z) \quad \forall n \in \mathbb{N}. \tag{2.16}$$

Taking the limit when n tends to infinity, we have

$$\lim_{n\to\infty} A(u_n) \le f(z). \tag{2.17}$$

From (2.8), (2.12), and (2.17), we have

$$f(z) = \alpha. \tag{2.18}$$

Since $z \in S(u_n)$ and $u_n \in S(u)$, by $(*)$, we obtain $z \in S(u)$. Suppose that $v_1 \in S(z)$ and $v_1 \ne z$. Then $f(v_1) < f(z)$ or by (2.18), $f(v_1) < \alpha$. Since $v_1 \in S(z)$, $z \in S(u_n)$ and $u_n \in S(u)$, by $(*)$, we have $S(z) \subseteq S(u_n) \subseteq S(u)$. Hence $v_1 \in S(u_n)$ and $v_1 \in S(u)$. Thus

$$A(u_n) \le f(v_1) \quad \forall n \in \mathbb{N}. \tag{2.19}$$

Taking the limit when n tends to infinity, we get

$$\alpha \le f(v_1). \tag{2.20}$$

This is in contradiction with $f(v_1) < \alpha$. Hence $S(z) = \{z\}$. But, by (2.3) and hypothesis of a function f in theorem there exists $y \in X$ such that $y \ne z$ and $\{y,z\} \subseteq S(z)$. This is a contradiction. This completes the proof. □

Remark 2.6. Theorem 2.5 is a generalization of Takahashi's minimization theorem [4].

Theorem 2.7. *Let (X,d) be left k-sequentially complete quasi-metric space such that for each $x \in X$, the mapping $u \to d(x,u)$ is a lower semicontinuous on X. Let $f : X \to (-\infty,\infty]$ be a proper weak lower semicontinuous function bounded from below. Assume that there exists a selfmapping T of X such that*

$$f(Tx)+d(x,Tx) \le f(x) \quad \forall x \in X. \tag{2.21}$$

Then T has a fixed point in X.

Proof. Since f is proper, there exists $v \in X$ such that $f(v) < \infty$. Put

$$Z = \{x \in X \mid f(x) \le f(v)\}. \tag{2.22}$$

Then, since f is weak lower semicontinuous, Z is closed. So Z is left k-sequentially complete. Let $x \in Z$. Then, Since

$$f(Tx)+d(x,Tx) \le f(x) \le f(v). \tag{2.23}$$

So Z is invariant under T. Assume that $Tx \ne x$ for every $x \in Z$. Then by Theorem 2.5, there exists $u \in Z$ such that $f(u) = \inf_{x\in X} f(x)$. Since $f(Tu)+d(u,Tu) \le f(u)$ and $f(u) = \inf_{x\in Z} f(x)$, we have $f(Tu) = f(u) = \inf_{x\in Z} f(x)$ and $d(u,Tu) = 0$. Hence $Tu = u$. This is a contradiction. Therefore T has a fixed point u in Z. This completes the proof. □

Remark 2.8. Theorem 2.7 is a generalization of Caristi's fixed point theorem [1].

The following theorem is a generalization of Ekeland's ε-variational principle [3].

Theorem 2.9. *Let (X,d) be left k-sequentially complete quasi-metric space such that for each $x \in X$ the mapping $u \to d(x,u)$ is a lower semicontinuous on X. Let $f : X \to (-\infty,\infty]$ be a proper weak lower semicontinuous function bounded from below. Then,*

(1) *for any $u \in X$ with $f(u) < \infty$, there exists $v \in X$ such that $f(v) \le f(u)$ and $f(w) > f(v) - d(v,w)$ for every $w \in X$ with $w \ne v$;*
(2) *for any $\varepsilon > 0$ and $u \in X$ with $f(u) < \inf_{x\in X} f(x)+\varepsilon$, there exists $v \in X$ such that $f(v) \le f(u)$, $d(u,v) \le 1$ and $f(w) > f(v) - \varepsilon d(v,w)$ for every $w \in X$ with $w \ne v$.*

Proof. (1) Let $u \in X$ be such that $f(u) < \infty$ and let

$$Y = \{x \in X \mid f(x) \le f(u)\}. \tag{2.24}$$

Then Y is nonempty and complete. We prove that there exists $v \in Y$ such that $f(w) > f(v) - d(v,w)$ for every $w \in X$ with $w \ne v$. If not, for every $x \in Y$, there exists $w \in X$ such that $w \ne x$ and $f(w)+d(x,w) \le f(x)$. Since $f(w) \le f(x) \le f(u)$, $w \in X$ is an element of Y. By Theorem 2.5, there exists $x_0 \in Y$ such that $f(x_0) = \inf_{x\in Y} f(x)$. For this $x_0 \in Y$, there exists $x_1 \in Y$ such that $x_0 \ne x_1$ and $f(x_1)+d(x_0,x_1) \le f(x_0)$.

Thus we have $f(x_0) = f(x_1)$ and $d(x_0, x_1) = 0$. Hence $x_0 = x_1$. This is a contradiction. Therefore (1) holds.

(2) Put

$$Z = \{x \in X \mid f(x) \le f(u) - \varepsilon d(u,x)\}. \tag{2.25}$$

Then Z is nonempty and complete. Since $\varepsilon d(u,x)$ is a quasi metric, as in the proof of (1), we have that there exists $v \in Z$ such that $f(w) > f(v) - \varepsilon d(v,w)$ for every $w \in X$ with $w \ne v$. Since $v \in Z$, we have $f(v) \le f(u) - \varepsilon d(u,v) \le f(u)$ and

$$d(u,v) \le \frac{1}{\varepsilon}[f(u) - f(v)] \le \frac{1}{\varepsilon}\Big[f(u) - \inf_{x\in X} f(x)\Big] \le \frac{1}{\varepsilon} \cdot \varepsilon = 1. \tag{2.26}$$

This completes the proof of (2). □

Remark 2.10. Theorem 2.9 is a generalization of Ekeland's ε-variational principle in [3].

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References

[1] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[2] L. Ćirić, *Semi-continuous mappings and fixed point theorems in quasi-metric spaces*, Publ. Math. Debrecen **54** (1999), no. 3-4, 251–261.

[3] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 3, 443–474.

[4] W. Takahashi, *Minimization theorems and fixed point theorems*, Nonlinear Analysis and Mathematical Economics (T. Maruyama, ed.), vol. 829, (1993), pp. 175–191.

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