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ON A WEAK FORM OF WEAK QUASI-CONTINUITY

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A weak form of weak quasi-continuity, which we call subweak quasi-continuity, is introduced. It is shown that subweak quasi-continuity is strictly weaker than weak quasi-continuity. Subweak quasi-continuity is used to strengthen several results in the literature concerning weak quasi-continuity. Specifically, results concerning the graph, graph function, and restriction of a weakly quasi-continuous function are extended slightly. Also, a result concerning weakly quasi-continuous retractions is strengthened.

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1. Introduction. Weakly quasi-continuous functions were introduced by Popa and Stan [9]. Recently, weak quasi-continuity has been developed further by Noiri [5, 6] and Park and Ha [8]. Due to a result by Noiri [5], weak quasi-continuity is equivalent to the weak semicontinuity developed by Arya and Bhamini [1]. The purpose of this note is to introduce the concept of subweak quasi-continuity, which we define in terms of a base for the topology on the codomain. We establish that this condition is strictly weaker than weak quasi-continuity and we use it to strengthen some of the results in the literature concerning weak quasi-continuity. For example, we show that the graph of a subweakly quasi-continuous function with a Hausdorff codomain is semiclosed. We also show that, if the graph function is subweakly quasi-continuous with respect to the usual base for the product space, then the function itself is weakly quasi-continuous, and that, if a function is subweakly quasi-continuous with respect to the base $\mathcal{B}$, then the restriction to a preopen set is subweakly quasi-continuous with respect to the same base. These results strengthen slightly the comparable results for weakly quasi-continuous functions. Finally, we extend a result concerning weakly quasi-continuous retractions and investigate some of the basic properties of subweakly quasi-continuous functions.

2. Preliminaries. The symbols X and Y denote topological spaces with no separation axioms assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A set A is semiopen (preopen, α-open) provided that $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ ($A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$, $A \subseteqq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). A set is semiclosed (preclosed, α-closed) provided that its complement is semiopen (preopen, α-open). The collection of all semiopen sets in a space X is denoted by $\mathrm{SO}(X)$ and the collection of all semiopen sets in X containing a fixed point x is denoted by $\mathrm{SO}(X,x)$. The intersection of all semiclosed sets containing a set A is called the semiclosure of A and denoted by $\mathrm{sCl}(A)$. The semi-interior of a set A, denoted by $\mathrm{sInt}(A)$, is the union of all semiopen sets contained in A. The preclosure of A, denoted by $\mathrm{pCl}(A)$, is the intersection of all preclosed sets containing A. Finally, if an operator is used with respect to a proper subspace, a subscript is added to the operator. Otherwise, it is assumed that the operator refers to the entire space.

Definition 2.1 (Popa and Stan [9]). A function $f : X \to Y$ is said to be weakly quasi-continuous if for every $x \in X$, every open set U in X containing x, and every open set V in Y containing $f(x)$, there exists a nonempty open set W in X such that $W \subseteq U$ and $f(W) \subseteq \mathrm{Cl}(V)$.

Definition 2.2 (Arya and Bhamini [1]). A function $f : X \to Y$ is said to be weakly semicontinuous if for every $x \in X$ and every open set V in Y containing $f(x)$, there exists $U \in \mathrm{SO}(X,x)$ for which $f(U) \subseteq \mathrm{Cl}(V)$.

The following result by Noiri [5] shows that weak quasi-continuity and weak semicontinuity are equivalent.

Theorem 2.3 (Noiri [5, Theorem 4.1]). *A function $f : X \to Y$ is weakly quasi-continuous if and only if for every $x \in X$ and every open set V containing $f(x)$, there exists $U \in \mathrm{SO}(X,x)$ for which $f(U) \subseteq \mathrm{Cl}(V)$.*

Definition 2.4. A function $f : X \to Y$ is said to be subweakly continuous (Rose [10]) (subalmost weakly continuous (Baker [2])) if there is an open base $\mathcal{B}$ for the topology on Y such that $\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ ($\mathrm{pCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$) for every $V \in \mathcal{B}$.

3. Subweakly quasi-continuous functions.

The following characterization of weak quasi-continuity is due to Noiri [5].

Theorem 3.1 (Noiri [5, Theorem 4.3(d)]). *A function $f : X \to Y$ is weakly quasi-continuous if and only if $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every open set V in Y.*

We define a function $f : X \to Y$ to be subweakly quasi-continuous provided that there is an open base $\mathcal{B}$ for the topology on Y for which $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathcal{B}$. Obviously, weak quasi-continuity implies subweak quasi-continuity. The following example shows that these concepts are not equivalent.

Example 3.2. Let $X = \mathbb{R}$ have the usual topology and $Y = X$ have the discrete topology. The identity mapping $f : X \to Y$ is subweakly quasi-continuous with respect to the base consisting of the singleton sets in Y. However, f is not weakly quasi-continuous because for $V = (0,1) \cup (1,2)$, $\mathrm{sCl}(f^{-1}(V)) \nsubseteq f^{-1}(\mathrm{Cl}(V))$.

Since $\mathrm{sCl}(A) = A \cup \mathrm{Int}(\mathrm{Cl}(A))$ for every set A, we have the following characterization of subweak quasi-continuity.

Theorem 3.3. *A function $f : X \to Y$ is subweakly quasi-continuous if and only if there is an open base $\mathcal{B}$ for the topology on Y for which $\mathrm{Int}(\mathrm{Cl}(f^{-1}(V))) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \subseteq \mathcal{B}$.*

Since $\mathrm{sCl}(A) \subseteq \mathrm{Cl}(A)$ for every set A, obviously, subweak continuity implies subweak quasi-continuity. The following example shows that the converse implication does not hold.

Example 3.4. Let $X = [1/2, 3/2]$ have the usual relative topology, $Y = \{0,1\}$ have the discrete topology, and let $f : X \to Y$ be the greatest integer function. Kar and Bhattacharya [3] showed that f is weakly quasi-continuous (their term is weakly semicontinuous) but not weakly continuous. Obviously, the function f is also not subweakly continuous.

The following two examples establish that subweak quasi-continuity is independent of subalmost weak continuity.

Example 3.5. Let X be an indiscrete space with at least two elements and let $Y = X$ have the discrete topology. Since $\mathrm{pCl}(\{x\}) = \{x\}$ for every $x \in X$, the identity mapping $f : X \to Y$ is subalmost weakly continuous with respect to the base consisting of the singleton sets in Y. However, since singleton sets in X are dense, f is not subweakly quasi-continuous.

Example 3.6. Let $X = \{a,b,c\}$ have the topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$ and $Y = X$ have the discrete topology. Let $f : X \to Y$ be the identity mapping. The function f is not subalmost weakly continuous, since any base for Y must include $V = \{a\}$ and $\mathrm{pCl}(f^{-1}(V)) \not\subseteq f^{-1}(\mathrm{Cl}(V))$. However, f is subweakly quasi-continuous with respect to the base of singleton subsets of Y.

4. Graph related properties. Recall that the graph of a function $f : X \to Y$ is the subspace $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Park and Ha [8] proved that the graph of a weakly quasi-continuous function with a Hausdorff codomain is semiclosed. We show that weak quasi-continuity can be replaced by subweak quasi-continuity.

Theorem 4.1. *If $f : X \to Y$ is subweakly quasi-continuous and Y is Hausdorff, then the graph of f, $G(f)$, is semiclosed.*

Proof. Let $\mathscr{B}$ be an open base for Y such that $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathscr{B}$. Let $(x,y) \in X \times Y - G(f)$. Since $y \neq f(x)$, there exists disjoint open sets V and W with $f(x) \in W$, $y \in V$, and $V \in \mathscr{B}$. Then $x \notin f^{-1}(\mathrm{Cl}(V))$, and, since $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$, $x \notin \mathrm{sCl}(f^{-1}(V))$. Therefore $(x,y) \in (X - \mathrm{sCl}(f^{-1}(V))) \times V \subseteq X \times Y - G(f)$. Since $\mathrm{sCl}(f^{-1}(V))$ is semiclosed, $X - \mathrm{sCl}(f^{-1}(V))$ is semiopen. Since finite products of semiopen sets are semiopen, $(X - \mathrm{sCl}(f^{-1}(V))) \times V$ is semiopen. Finally, since unions of semiopen sets are semiopen, it follows that $X \times Y - G(f)$ is semiopen and that $G(f)$ is semiclosed. □

Corollary 4.2 (Park and Ha [8, Corollary 4.2])**.** *If $f : X \to Y$ is weakly quasi-continuous and Y is Hausdorff, then the graph of f, $G(f)$, is semiclosed.*

By the graph function of a function $f : X \to Y$ we mean the function $g : X \to X \times Y$ given by $g(x) = (x, f(x))$ for every $x \in X$.

Theorem 4.3. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function and let $\mathcal{B}$ be an open base for σ. Let $\mathcal{C} = \{U \times V : U \in \tau,\ V \in \mathcal{B}\}$. The function f is subweakly quasi-continuous with respect to the base $\mathcal{B}$ if and only if the graph function of f, $g : X \to X \times Y$, is subweakly quasi-continuous with respect to the base $\mathcal{C}$.*

Proof. Assume that $f : (X,\tau) \to (Y,\sigma)$ is subweakly quasi-continuous with respect to the base $\mathcal{B}$ for σ. Let $U \times V \in \mathcal{C}$, where $U \in \tau$ and $V \in \mathcal{B}$. Then $\mathrm{sCl}(g^{-1}(U \times V)) = \mathrm{sCl}(U \cap f^{-1}(V)) \subseteq \mathrm{sCl}(U) \cap \mathrm{sCl}(f^{-1}(V)) \subseteq \mathrm{Cl}(U) \cap f^{-1}(\mathrm{Cl}(V)) = g^{-1}(\mathrm{Cl}(U) \times \mathrm{Cl}(V)) = g^{-1}(\mathrm{Cl}(U \times V))$. Thus g is subweakly quasi-continuous with respect to the base $\mathcal{C}$.

Assume that $g : (X,\tau) \to X \times Y$ is subweakly quasi-continuous with respect to the base $\mathcal{C}$ for $X \times Y$. If $V \in \mathcal{B}$, then $\mathrm{sCl}(f^{-1}(V)) = \mathrm{sCl}(g^{-1}(X \times V)) \subseteq g^{-1}(\mathrm{Cl}(X \times V)) = g^{-1}(X \times \mathrm{Cl}(V)) = f^{-1}(\mathrm{Cl}(V))$. Therefore, f is subweakly quasi-continuous with respect to the base $\mathcal{B}$. □

In Theorem 4.3, if we take $\mathcal{B}$ to be σ, the topology on Y, then we have the following result.

Corollary 4.4. *If the graph function $g : X \to X \times Y$ of a function f is subweakly quasi-continuous with respect to the usual base for the product space, then the function f is weakly quasi-continuous.*

Corollary 4.5 (Noiri [5, The "only if" part of Theorem 6.3.4]). *If the graph function $g : X \to X \times Y$ of a function f is weakly quasi-continuous, then the function f is weakly quasi-continuous.*

5. Additional properties

Definition 5.1 (Kar and Bhattacharya [4]). A space X is said to be semi-T_1 provided that for every pair of distinct points x and y in X there exist sets $U \in \mathrm{SO}(X,x)$ and $V \in \mathrm{SO}(X,y)$ such that $y \notin U$ and $x \notin V$.

Theorem 5.2. *If Y is Hausdorff and $f : X \to Y$ is a subweakly quasi-continuous injection, then X is semi-T_1.*

Proof. Let x_1 and x_2 be distinct points in X and let $\mathcal{B}$ be an open base for Y such that $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff and $f(x_1) \neq f(x_2)$, there exist disjoint open sets U and V in Y with $f(x_1) \in U$ and $f(x_2) \in V$, and $V \in \mathcal{B}$. Then, since $f(x_1) \notin \mathrm{Cl}(V)$, we have $x_1 \in X - f^{-1}(\mathrm{Cl}(V)) \subseteq X - \mathrm{sCl}(f^{-1}(V))$ which is semiopen and does not contain x_2. Therefore X is semi-T_1. □

The function in Example 3.6 is a subweakly quasi-continuous injection with a Hausdorff codomain and a non-T_1-domain. Therefore, the conclusion that X is semi-T_1 in Theorem 5.2 cannot be strengthened to T_1.

Since the restriction of the function f in Example 3.6 to the set $A = \{a,c\}$ is not subweakly quasi-continuous, we see that the restriction of a subweakly quasi-continuous function can fail to be subweakly quasi-continuous. Noiri [5] proved that the restriction of weakly quasi-continuous function to an open set is weakly quasi-continuous and Arya and Bhamini [1] extended this result to α-open sets. Finally, Park and Ha [8] extended the result further to preopen sets. In what follows, we establish the analogous result for subweakly quasi-continuous functions.

Theorem 5.3. *If $f : X \to Y$ is subweakly quasi-continuous with respect to the base $\mathcal{B}$ for Y and A is a preopen set in X, then $f|_A : A \to Y$ is subweakly quasi-continuous with respect to the base $\mathcal{B}$.*

Proof. Let $V \in \mathcal{B}$, then using (Noiri [7, Lemma 3.3]) we see that $\mathrm{sCl}_A(f|_A^{-1}(V)) = A \cap \mathrm{sCl}(f|_A^{-1}(V)) = A \cap \mathrm{sCl}(f^{-1}(V) \cap A) \subseteq A \cap \mathrm{sCl}(f^{-1}(V)) \subseteq A \cap f^{-1}(\mathrm{Cl}(V)) = f|_A^{-1}(\mathrm{Cl}(V))$. Therefore, $f|_A : A \to Y$ is subweakly quasi-continuous with respect to the base $\mathcal{B}$. □

In Theorem 5.3, if we let $\mathcal{B}$ be the topology, then we have the following result.

Corollary 5.4 (Park and Ha [8, Theorem 3.8])**.** *If $f : X \to Y$ is weakly quasi-continuous and A is an preopen set in X, then $f|_A : A \to Y$ is weakly quasi-continuous.*

Theorem 5.5. *If $f : X \to Y$ is subweakly quasi-continuous and A is an open set in Y containing $f(X)$, then $f : X \to A$ is subweakly quasi-continuous.*

Proof. Let $\mathcal{B}$ be an open base for Y for which $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathcal{B}$. Then $\mathcal{C} = \{V \cap A : V \in \mathcal{B}\}$ is an open base for the relative topology on A. Let $V \cap A \in \mathcal{C}$, where $V \in \mathcal{B}$. Then $\mathrm{sCl}(f^{-1}(V \cap A)) = \mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V)) = f^{-1}(\mathrm{Cl}(V) \cap A)$. The proof is completed by establishing that $\mathrm{Cl}(V) \cap A \subseteq \mathrm{Cl}_A(V \cap A)$.

Let $y \in \mathrm{Cl}(V) \cap A$ and let $W \subseteq A$ be open in A with $y \in W$. Since A is open in Y, W is open in Y. Because $y \in \mathrm{Cl}(V)$, $W \cap V \neq \varnothing$. Therefore $W \cap (V \cap A) \neq \varnothing$, which proves that $y \in \mathrm{Cl}_A(V \cap A)$. Thus $\mathrm{Cl}(V) \cap A \subseteq \mathrm{Cl}_A(V \cap A)$.

Now, it follows that $f : X \to A$ is subweakly quasi-continuous. □

Park and Ha [8] defined a function $f : X \to A$, where $A \subseteq X$, to be a weakly quasi-continuous retraction provided that f is weakly quasi-continuous and $f|_A$ is the identity on A. It is then proved (Park and Ha [8, Theorem 3.15]) that, if $f : X \to A$ is a weakly quasi-continuous retraction and X is Hausdorff, then A is semiclosed in X. We prove the following comparable result for subweakly quasi-continuous functions.

Theorem 5.6. *Let $A \subseteq X$ and let $f : X \to X$ be a subweakly quasi-continuous function such that $f(X) = A$ and $f|_A$ is the identity on A. If X is Hausdorff, then A is semiclosed.*

Proof. Assume A is not semiclosed. Let $x \in \mathrm{sCl}(A) - A$. Let $\mathcal{B}$ be an open base for the topology on X such that $\mathrm{sCl}(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathcal{B}$. Since $x \notin A$, $x \neq f(x)$. Because X is Hausdorff, there exist disjoint open sets V and W such that $x \in V$, $f(x) \in W$, and $V \in \mathcal{B}$. Let $U \in \mathrm{SO}(X,x)$. Then $x \in U \cap V$, which is semiopen in X (Noiri [7]). Since $x \in \mathrm{sCl}(A)$, $(U \cap V) \cap A \neq \varnothing$. So there exists $y \in (U \cap V) \cap A$. Since $y \in A$, $f(y) = y$ and therefore $y \in f^{-1}(V)$. Thus $U \cap f^{-1}(V) \neq \varnothing$ and we see that $x \in \mathrm{sCl}(f^{-1}(V))$. However, $f(x) \in W$, which is open and disjoint from V. Hence $f(x) \notin \mathrm{Cl}(V)$ or $x \notin f^{-1}(\mathrm{Cl}(V))$, which contradicts the assumption that f is subweakly quasi-continuous. □

Lemma 5.7. *If $A \subseteq Y$ and $f : X \to A$ is weakly quasi-continuous, then $f : X \to Y$ is weakly quasi-continuous.*

Proof. If V is an open set in Y, then $\mathrm{sCl}(f^{-1}(V)) = \mathrm{sCl}(f^{-1}(V \cap A)) \subseteq f^{-1}(\mathrm{Cl}_A(V \cap A)) = f^{-1}(A \cap \mathrm{Cl}(V \cap A)) = f^{-1}(\mathrm{Cl}(V \cap A)) \subseteq f^{-1}(\mathrm{Cl}(V))$. □

Thus a weakly quasi-continuous retraction satisfies the hypothesis of Theorem 5.6 and we have the following corollary.

Corollary 5.8 (Park and Ha [8, Theorem 3.15])**.** *If $f : X \to A$, where $A \subseteq X$, is a weakly quasi-continuous retraction and X is Hausdorff, then A is semiclosed.*

Theorem 5.9. *Let Y be a Hausdorff space, $f_1 : X \to Y$ continuous, and $f_2 : X \to Y$ subweakly quasi-continuous. Then $\{x \in X : f_1(x) = f_2(x)\}$ is semiclosed.*

Proof. Let $A = \{x \in X : f_1(x) = f_2(x)\}$ and let $x \in X - A$. Let $\mathcal{B}$ be an open base for the topology on Y for which $\mathrm{sCl}(f_2^{-1}(V)) \subseteq f_2^{-1}(\mathrm{Cl}(V))$ for every $V \in \mathcal{B}$. Since Y is Hausdorff and $f_1(x) \neq f_2(x)$, there exist disjoint open sets V and W in Y for which $f_1(x) \in V$, $f_2(x) \in W$, and $V \in \mathcal{B}$. Since $f_2(x) \notin \mathrm{Cl}(V)$, we have $x \in X - f_2^{-1}(\mathrm{Cl}(V)) \subseteq X - \mathrm{sCl}(f_2(V))$. Therefore $x \in f_1^{-1}(V) \cap (X - \mathrm{sCl}(f_2^{-1}(V))) \subseteq X - A$. Since $f_1^{-1}(V)$ is open, $X - \mathrm{sCl}(f_2^{-1}(V))$ is semiopen, and the intersection of an open set and a semiopen set is semiopen (Noiri [7]), we see that $X - A$ is semiopen and that A is semiclosed. □

Corollary 5.10. *Let Y be Hausdorff, $f_1 : X \to Y$ continuous, and $f_2 : X \to Y$ subweakly quasi-continuous. If f_1 and f_2 agree on a dense subset of X, then $f_1 = f_2$.*

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