IJMMS 31:6 (2002) 353–358
PII. S0161171202007883
http://ijmms.hindawi.com

SOLVING SOME NONLINEAR EQUATIONS BY SUCCESSIVE APPROXIMATIONS

ALBERT K. KALINDE

Received 10 June 2001

The iterations scheme generated by an infinite sequence of operators satisfying some contractive conditions in a complete metric space is used to solve some integral equations of Hammerstein type.

2000 Mathematics Subject Classification: 47H10, 45G10, 47J25.

1. Introduction. Iterations processes are powerful tools for solving both differential and integral equations in a complete metric space. One of the generalizations of the Picard iterations scheme consists of iterations generated by an infinite sequence (T_n) of operators T_n and described by

$$x_0 \in X, \qquad x_{n+1} = T_n(x_n), \tag{1.1}$$

where X is a complete metric space and T_n a selfmap of X.

This type of schemes was considered and studied by some authors using various contractive conditions (see [1] and the references therein).

The consideration of (1.1) is motivated by its application to Hammerstein integral equations. Its use gives rise to new sufficient conditions for the existence of solutions of these equations. For that purpose, (1.1) is formulated as follows:

$$x_0 \in X, \qquad X_{n+1} = T_n f(x_n), \tag{1.2}$$

where both T_n and f are selfmaps of X. The contractive condition to be satisfied in our paper is given by

$$d(T_m fx, T_n fy) \le \max\{\gamma d(x,y), ad(x,T_m fx) + bd(y,T_n fy)\}, \tag{1.3}$$

for any $m,n \in \mathbb{N}$ and for any $x,y \in X$, where $0 \le \gamma < 1$ and a, b are given real numbers such that $0 \le a$, $b < 1$ with $a+b < 1$.

A slightly weakened contractive condition considered is described by

$$d(T_n fx, T_{n+1} fy) \le \max\{\gamma d(x,y), ad(x,T_n fx) + bd(y,T_{n+1} fy)\}, \tag{1.4}$$

for all $x,y \in X$ and all n sufficiently large, where γ, a, and b are the numbers appearing in (1.3).

The type of Hammerstein integral equations we consider in this paper is in the form

$$u(x) + \int_\Omega k(x,y) f(y,u(y))\,dy = 0, \tag{1.5}$$

where Ω is a measurable subset of $\mathbb{R}^N$ (or $\mathbb{C}^N$), $N \geqslant 1$ and $k(x,y)$, $f(x,t)$ are real- or complex-valued functions defined on $\Omega\times\Omega$ and on $\Omega\times\mathbb{R}$ (or on $\Omega\times\mathbb{C}$), respectively and measurable in both variables. We are looking for solutions of (1.5) in the Banach space $L^1(\Omega)$ of real- or complex-valued functions u on Ω that are Lebesgue integrable on Ω, equipped with its usual norm $\|\cdot\|_{L^1(\Omega)}$ given by

$$\|u\|_{L^1(\Omega)} = \int_\Omega |u(x)|\,dx. \tag{1.6}$$

2. Hammerstein integral equations

Theorem 2.1. *Let Ω, $f(x,t)$, and $k(x,y)$ be as described above and such that*

(1) *$f(x,u(x)) \in L^1(\Omega)$ for all $u \in L^1(\Omega)$,*

(2) *for every $u \in L^1(\Omega)$, there exists a nonnegative function $\beta \in L^1(\Omega)$ such that for almost all $x \in \Omega$, $\int_\Omega |k(x,y)u(y)|dy \le \beta(x)$,*

(3) *there exists a real number $0 \le h < 1/3$ such that*

$$\int_\Omega \left|k(x,y)[u(y)-v(y)]\right|dy \le h\left|u(x)-\int_\Omega k(x,y)v(y)\,dy\right| \tag{2.1}$$

for all $u,v \in L^1(\Omega)$ and for almost all x,y in Ω,

(4) *$|f(y,u(y))-v(y)| \le |u(y)-v(y)|$ for all $u,v \in L^1(\Omega)$.*

Then, (1.5) has a unique solution in $L^1(\Omega)$.

Proof. For every $u \in L^1(\Omega)$, we may set

$$\begin{aligned} fu(x) &= f(x,u(x)),\\ Kfu(x) &= \int_\Omega k(x,y)f(y,u(y))\,dy. \end{aligned} \tag{2.2}$$

By conditions (1) and (2), Kf defines an operator from $L^1(\Omega)$ into itself. From condition (3), we deduce that

$$\|Ku-Kv\|_{L^1(\Omega)} \le h\|u-Kv\|_{L^1(\Omega)}. \tag{2.3}$$

For all natural number $n \ge 1$, we may set

$$K^n fu(x) = K(K^{n-1}fu(x)) = \int_\Omega k(x,y)K^{n-1}fu(y)\,dy \tag{2.4}$$

with the convention that $K^0 = I$.

First we prove by mathematical induction on $n \in \mathbb{N}_0$, with $\mathbb{N}_0 = \mathbb{N}\setminus\{0\}$ that for all $u,v \in L^1(\Omega)$ and all $n,m \in \mathbb{N}_0$ we have

$$||K^n fu - K^m fv||_{L^1(\Omega)} \le h^n||fu - K^m fv||_{L^1(\Omega)}. \tag{2.5}$$

Indeed, for $n = 1$ we have by (2.3),

$$||Kfu-K^m fv||_{L^1(\Omega)} = ||K(fu-K^{m-1}fv)||_{L^1(\Omega)} \le h||fu-K^m fv||_{L^1(\Omega)}. \tag{2.6}$$

Assume that for $n = r$ we have

$$||K^r fu - K^m fv||_{L^1(\Omega)} \le h^r||fu-K^m fv||_{L^1(\Omega)}, \tag{2.7}$$

then, for $n = r+1$ we get

$$\|K^{r+1}fu - K^m fv\|_{L^1(\Omega)} \le h\|K^r fu - K^m fv\|_{L^1(\Omega)} \le h^{r+1}\|fu - K^m fv\|_{L^1(\Omega)} \tag{2.8}$$

which establishes the claim.

But conditions (1) and (4) imply that

$$\|fu - K^m fv\|_{L^1(\Omega)} \le \|u - K^m fv\|_{L^1(\Omega)}. \tag{2.9}$$

Furthermore, we know that

$$\|u - K^m fv\|_{L^1(\Omega)} \le \|u - K^n fu\|_{L^1(\Omega)} + \|K^n fu - K^m fv\|_{L^1(\Omega)}, \tag{2.10}$$

as $0 < h < 1$, it is clear that $h^n \le h$ and hence, we get

$$\|K^n fu - K^m fv\|_{L^1(\Omega)} \le h\|u - K^n fu\|_{L^1(\Omega)} + h\|K^n fu - K^m fv\|_{L^1(\Omega)}, \tag{2.11}$$

which finally yields

$$\|K^n fu - K^m fv\|_{L^1(\Omega)} \le \frac{h}{1-h}\|u - K^n fu\|_{L^1(\Omega)} + \frac{h}{1-h}\|v - K^m fv\|_{L^1(\Omega)}. \tag{2.12}$$

Condition (1.3) is, therefore, satisfied, with $a = b = h/(1-h)$ and

$$\frac{h}{1-h} + \frac{h}{1-h} = \frac{2h}{1-h} < 1, \quad \text{because } 3h < 1. \tag{2.13}$$

Now, for arbitrary $u_0 \in L^1(\Omega)$, we may consider the approximation process (1.2) in the following form:

$$u_0 \in L^1(\Omega), \qquad u_{n+1} = K^n f(u_n), \tag{2.14}$$

and show that it converges. Indeed, by mathematical induction on $n \in \mathbb{N}_0$, we establish that

$$\|u_n - u_{n+1}\|_{L^1(\Omega)} \le \left(\max\left\{\gamma, \frac{h}{1-2h}\right\}\right)^n \|u_0 - u_1\|_{L^1(\Omega)}, \tag{2.15}$$

where γ is given in (1.3). Indeed, for $n = 1$ we have by (2.14) and by (2.12) that

$$\begin{aligned} \|u_1 - u_2\|_{L^1(\Omega)} &= \|K^0 fu_0 - Kfu_1\|_{L^1(\Omega)} \\ &\le \frac{h}{1-h}\|u_0 - u_1\|_{L^1(\Omega)} + \frac{h}{1-h}\|u_1 - u_2\|_{L^1(\Omega)}, \end{aligned} \tag{2.16}$$

from which we deduce that

$$\|u_1 - u_2\|_{L^1(\Omega)} \le \frac{h}{1-2h}\|u_0 - u_1\|_{L^1(\Omega)}, \tag{2.17}$$

hence (2.15) is valid for $n = 1$.

Assume now that (2.15) is true for $n = r$, we prove it for $n = r+1$. Indeed, using (2.14) and (2.12) again we get

$$\begin{aligned} \|u_{r+1} - u_{r+2}\|_{L^1(\Omega)} &= \|K^r f u_r - K^{r+1} f u_{r+1}\|_{L^1(\Omega)} \\ &\le \frac{h}{1-h}\|u_r - K^r f u_r\|_{L^1(\Omega)} + \frac{h}{1-h}\|u_{r+1} - K^{r+1} f u_{r+1}\|_{L^1(\Omega)}, \end{aligned} \tag{2.18}$$

from which we deduce, after applying (2.14) again, that

$$\|u_{r+1} - u_{r+2}\|_{L^1(\Omega)} \le \frac{h}{1-2h}\|u_r - u_{r+1}\|_{L^1(\Omega)}. \tag{2.19}$$

Then, the induction hypothesis applied to the right-hand side yields that (2.15) is valid for $n = r+1$ and this establishes the claim.

Now set $\alpha = \max\{\gamma, h/(1-2h)\}$, it is clear that $0 \le \alpha < 1$.

We show that (u_n) is a Cauchy sequence. Indeed, for all $p \in \mathbb{N}$ we obtain by applying (2.15) that

$$\begin{aligned} \|u_n - u_{n+p}\|_{L^1(\Omega)} &\le \|u_n - u_{n+1}\|_{L^1(\Omega)} + \cdots + \|u_{n+p-1} - u_{n+p}\|_{L^1(\Omega)} \\ &\le \alpha^n (1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1})\|u_0 - u_1\|_{L^1(\Omega)} \\ &\le \frac{\alpha^n}{1-\alpha}\|u_0 - u_1\|_{L^1(\Omega)}. \end{aligned} \tag{2.20}$$

This implies that (u_n) is a Cauchy sequence in $L^1(\Omega)$ and hence converges to, say $w \in L^1(\Omega)$.

We prove that $K^n f w = w$ for all $n \in \mathbb{N}_0$. For arbitrary $r \in \mathbb{N}_0$, we get

$$\|w - K^r f w\|_{L^1(\Omega)} \le \|w - u_n\|_{L^1(\Omega)} + \|u_n - K^n f u_n\|_{L^1(\Omega)} + \|K^n f u_n - K^r f w\|_{L^1(\Omega)}. \tag{2.21}$$

Applying (2.12) and (2.14) finally yields

$$\|w - K^r f w\|_{L^1(\Omega)} \le \frac{1-h}{1-2h}\|w - u_n\|_{L^1(\Omega)} + \frac{1}{1-2h}\|u_n - u_{n+1}\|_{L^1(\Omega)}. \tag{2.22}$$

For n tending to infinity, the right-hand side converges to 0 and this is valid for any $r \in N_0$. Hence, the claim is true and, therefore, (1.5) has a unique solution. □

Theorem 2.2. *Let Ω be a measurable set in $\mathbb{R}^N$ (or in $\mathbb{C}^N$) with $N \geqslant 1$. Let $f(x,t)$ be a function defined on $\Omega \times \mathbb{R}$ (or on $\Omega \times \mathbb{C}$), that is, real- or complex-valued and measurable in both variables. Let $fk(x,y)$ be a real- or complex-valued function defined on $\Omega \times \Omega$, that is, the limit almost everywhere on $\Omega \times \Omega$ of a sequence $(k_n(x,y))$ of real- or complex-valued functions $k_n(x,y)$ that are measurable in both variables on $\Omega \times \Omega$. Assume that these functions are such that*

(i) *$f(x,u(x)) \in L^1(\Omega)$ for every $u \in L^1(\Omega)$,*

(ii) *for every $u \in L^1(\Omega)$, there exists a nonnegative function $\beta(x,y) \in L^1(\Omega \times \Omega)$ such that $|k_n(x,y)u(y)| \le \beta(x,y)$ a.e. on $\Omega \times \Omega$ for every $n \in \mathbb{N}$,*

(iii) *there exist real numbers* $0 \le a, b < 1$ *with* $a+b<1$*, such that*

$$(1+a)\int_\Omega |k_n(x,y)f(y,u(y))|\,dy + (1+b)\int_\Omega |k_{n+1}(x,y)f(y,v(y))|\,dy \le a|u(x)|+b|v(x)| \tag{2.23}$$

for all $u,v\in L^1(\Omega)$*, for almost all* $x\in\Omega$ *and for all* $n\in\mathbb{N}$ *sufficiently large. Then (1.5) has a unique solution in* $L^1(\Omega)$*.*

Proof. Let $u\in L^1(\Omega)$ be arbitrary. For every $n\in\mathbb{N}$ sufficiently large, as $k_n(x,y)$ is measurable in both variables, conditions (i) and (ii) imply, by Fubini's theorem applicable to $\beta(x,y)$, that the quantity K_nfu defined on Ω by

$$K_nfu(x)=\int_\Omega k_n(x,y)f(y,u(y))\,dy \tag{2.24}$$

is a real- or complex-valued function that belongs to $L^1(\Omega)$ and hence that K_nf is a selfmap of $L^1(\Omega)$. Since $\lim_{n\to\infty}k_n(x,y)=k(x,y)$ a.e. on $\Omega\times\Omega$, the Lebesgue dominated convergence theorem implies by condition (ii) that for every $x\in\Omega$,

$$\lim_{n\to\infty}K_nfu(x)=\lim_{n\to\infty}\int_\Omega k_n(x,y)f(y,u(y))\,dy=\int_\Omega k(x,y)f(y,u(y))\,dy. \tag{2.25}$$

Therefore, Kfu defined by

$$Kfu(x)=\int_\Omega k(x,y)f(y,u(y))\,dy \tag{2.26}$$

is a function that belongs to $L^1(\Omega)$. As $u\in L^1(\Omega)$ is arbitrary and that the sequence (K_nfu) converges pointwise to Kfu, hence the sequence (K_nf) of operators from $L^1(\Omega)$ into itself converges pointwise to the operator Kf also from $L^1(\Omega)$ into itself.

We now show that there exist $0\le a, b<1$ with $a+b<1$, such that

$$|K_nfu(x)-K_{n+1}fv(x)|\le a|u(x)-K_nfu(x)|+b|v(x)-K_{n+1}fv(x)| \tag{2.27}$$

a.e. on Ω for all $u,v\in L^1(\Omega)$ and all $n\in\mathbb{N}$ sufficiently large.

Assume that for all $0\le a,b<1$ with $a+b<1$, there exist $u_0,v_0\in L^1(\Omega)$ and $n_0\in\mathbb{N}$ sufficiently large such that

$$|K_{n_0}fu_0(x)-K_{n_0+1}fv_0(x)|>a|u_0(x)-K_{n_0}fu_0(x)|+b|v_0(x)-K_{n_0+1}fv_0(x)| \tag{2.28}$$

a.e. Then we have

$$|K_{n_0}fu_0(x)|+|K_{n_0+1}fv_0(x)|>a|u_0(x)|+b|v_0(x)| - a|K_{n_0}fu_0(x)|-b|K_{n_0+1}fv_0(x)| \tag{2.29}$$

a.e. Therefore, we have

$$a|u_0(x)|+b|v_0(x)|<(1+a)\int_\Omega |k_{n_0}(x,y)f(y,u_0(y))|\,dy + (1+b)\int_\Omega |k_{n_0+1}(x,y)f(y,v_0(y))|\,dy, \tag{2.30}$$

which is a contradiction to condition (iii). Hence, (2.27) is valid. Consequently, we deduce that for all n sufficiently large

$$||K_n f u - K_{n+1} f v||_{L^1(\Omega)} \le a||u - K_n f u||_{L^1(\Omega)} + b||v - K_{n+1} f v||_{L^1(\Omega)}. \tag{2.31}$$

Therefore, if we consider the iterations process

$$u_0 \in L^1(\Omega), \qquad u_{n+1} = K_n f(u_n), \tag{2.32}$$

by using the same argument as in the proof of Theorem 2.1, then the sequence (u_n) converges to say $w \in L^1(\Omega)$.

We prove that $\lim_{n\to\infty} K_n f w = w$ in $L^1(\Omega)$. Indeed, for n sufficiently large, we obtain

$$\begin{aligned} ||w - K_n f w||_{L^1(\Omega)} &\le ||w - u_{n+2}||_{L^1(\Omega)} + ||u_{n+2} - K_n f w||_{L^1(\Omega)} \\ &\le ||w - u_{n+2}||_{L^1(\Omega)} + ||K_{n+1} f u_{n+1} - K_n f w||_{L^1(\Omega)}. \end{aligned} \tag{2.33}$$

Then applying (2.31), we get

$$||w - K_n f w||_{L^1(\Omega)} \le ||w - u_{n+2}||_{L^1(\Omega)} + a||u_{n+1} - K_{n+1} f u_{n+1}||_{L^1(\Omega)} + b||w - K_n f w||_{L^1(\Omega)}, \tag{2.34}$$

from which we deduce that for n sufficiently large,

$$(1-b)||w - K_n f w||_{L^1(\Omega)} \le ||w - u_{n+2}||_{L^1(\Omega)} + a||u_{n+1} - u_{n+2}||_{L^1(\Omega)}. \tag{2.35}$$

This implies that $\lim_{n\to\infty} K_n f w = w$ in $L^1(\Omega)$. As the sequence $(K_n f)$ of operators converges pointwise to the operator Kf, hence

$$w = \lim_{n\to\infty} (K_n f w) = \lim_{n\to\infty} K_n f(w) = Kf(w), \tag{2.36}$$

and this shows that $w \in L^1(\Omega)$ is the unique solution of (1.5). This completes the proof. □

References

[1] B. E. Rhoades, *Fixed point theorems for some families of maps*, Indian J. Pure Appl. Math. **21** (1990), no. 1, 10–20.

Albert K. Kalinde: Department of Mathematics, University of Fort Hare, Private Bag X1314, Alice 5700, South Africa