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γ-SETS AND γ-CONTINUOUS FUNCTIONS

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We introduce a new class of sets, called γ-sets, and the notion of γ-continuity and investigate some properties and characterizations. In particular, γ-sets and γ-continuity are used to extend known results for semi-open sets and semi-continuity.

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1. Introduction. Let X, Y, and Z be topological spaces on which no separation axioms are assumed unless explicitly stated. Let S be a subset of X. The closure (resp., interior) of S will be denoted by $\mathrm{cl}\,S$ (resp., $\mathrm{int}\,S$). A subset S of X is called a semi-open set [2] (resp., α-set [4]) if $S \subset \mathrm{cl}(\mathrm{int}(S))$ (resp., $S \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(S)))$). The complement of a semi-open set (resp., α-set) is called semi-closed set (resp., α-closed set). The family of all semi-open sets (resp., α-sets) in X will be denoted by $\mathrm{SO}(X)$ (resp., $\alpha(X)$). A function $f : X \to Y$ is called semi-continuous [2] (resp., α-continuous [3]) if $f^{-1}(V) \in \mathrm{SO}(X)$ (resp., $f^{-1}(V) \in \alpha(X)$) for each open set V of Y. A function $f : X \to Y$ is called semi-open [2] (resp., α-open [3]) if for every semi-open (resp., α-open) set U in X, $f(U)$ is semi-open (resp., α-open) in Y.

A subset $M(x)$ of a space X is called a semi-neighborhood of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subset M(x)$. In [1], Latif introduced the notion of semi-convergence of filters and investigated some characterizations related to semi-open continuous functions. Now, we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in \mathrm{SO}(X) : x \in A\}$ and let $S_x = \{A \subset X : \exists \mu \subset S(x)$ such that μ is finite and $\cap\mu \subset A\}$. Then, S_x is called the semi-neighborhood filter at x. For any filter F on X, we say that F semi-converges to x if and only if F is finer than the semi-neighborhood filter at x.

2. γ-sets

Definition 2.1. Let (X,τ) be a topological space. A subset U of X is called a γ-set in X if whenever a filter F semi-converges to x and $x \in U$, $U \in F$.

The class of all γ-sets in X will be denoted by $\gamma(X)$. In particular, the class of all γ-sets induced by the topology τ will be denoted by γ_τ.

Remark 2.2. From the definition of semi-neighborhood filter and γ-set, we can easily say that every semi-open set is a γ-set, but the converse is always not true.

Example 2.3. Let X be the real number set with the usual topology. For each $x \in X$, since both $(a,x]$ and $[x,b)$ are semi-open sets containing x, where $a < x < b$, $\{x\}$ is an element of S_x. For any filter F on X, if F semi-converges to x and since F includes S_x, then x is a γ-set. But it is not semi-open.

Remark 2.4. In a topological space (X,τ), it is always true that

$$\tau \subset \alpha(X) \subset \mathrm{SO}(X) \subset \gamma(X). \tag{2.1}$$

Theorem 2.5. *Let (X,τ) be a topological space. The intersection of finitely many semi-open subsets in X is a γ-set.*

Proof. Let U_1 and U_2 be semi-open sets in X. For each $x \in U_1 \cap U_2$, we get $U_1 \cap U_2 \in S_x$. Thus, from the concept of the semi-convergence of filters, whenever every filter F semi-converges to x, $U_1 \cap U_2 \in F$. □

Definition 2.6. Let (X,τ) be a topological space. The γ-interior of a set A in X, denoted by $\mathrm{int}_\gamma(A)$, is the union of all γ-sets contained in A.

Theorem 2.7. *Let (X,τ) be a topological space and $A \subset X$.*
(a) $\mathrm{int}_\gamma(A) = \{x \in A : A \in S_x\}$.
(b) *A is γ-set if and only if $A = \mathrm{int}_\gamma(A)$.*

Proof. (a) For each $x \in \mathrm{int}_\gamma(A)$, there exists a γ-set U such that $x \in U$ and $U \subset A$. From the notion of γ-set, the subset U is in the semi-neighborhood filter S_x. Since S_x is a filter, $A \in S_x$. Conversely, let $x \in A$ and $A \in S_x$, then there exist $U_1 \cdots U_n \in S(x)$ such that $U = U_1 \cap \cdots \cap U_n \subset A$. By Theorem 2.5, U is a γ-set and $U \subset A$. Thus $x \in \mathrm{int}_\gamma(A)$.
(b) The proof is obvious. □

Theorem 2.8. *Let (X,τ) be a topological space. Then, the class $\gamma(X)$ of all γ-subsets in X is a topology on X.*

Proof. Since $\varnothing$ and X are semi-open, they are also γ-sets in X. Let $A,B \in \gamma(X)$, $x \in A \cap B$, and let F be a filter. Suppose the filter F semi-converges to x. Then A, $B \in F$ and since F is a filter, $(A \cap B) \in F$. Thus, $A \cap B$ is a γ-set.

For each $\alpha \in I$ let $A_\alpha \in \gamma(X)$ and $U = \cup A_\alpha$. For each $x \in U$ and for a filter F semi-converging to x there exists a subset A_α of U such that $x \in A_\alpha$, and since A_α is γ-set, it is obvious that $A_\alpha \in F$. Since F is a filter, U is an element of the filter F and thus $U = \cup A_\alpha$ is a γ-set.

In a topological space (X,τ), the class of all γ-sets induced by the topology τ will be denoted by (X,γ_τ). A subset B of X is called a γ-closed set if the complement of B is a γ-set. Thus, the intersection of any family of γ-closed sets is a γ-closed set and the union of finitely many γ-closed sets is a γ-closed set. □

Obviously, we obtain the following theorem by definition of the γ-set.

Theorem 2.9. *Let (X,τ) be a topological space. A set G is γ-closed if and only if whenever F semi-converges to x and $A \in F$, $x \in A$.*

Definition 2.10. Let (X,τ) be a topological space and $A \subset X$,

$$\mathrm{cl}_\gamma(A) = \{x \in X : A \cap U \neq \varnothing \ \forall U \in S_x\}. \tag{2.2}$$

We call $\mathrm{cl}_\gamma(A)$ the γ-closure of the set A.

Now we can get the following theorem.

Theorem 2.11. *Let (X,τ) be a topological space and let A be a subset of X. Then the following properties hold:*
(1) $A \subset \mathrm{cl}_\gamma(A)$;
(2) *A is γ-closed if and only if* $A = \mathrm{cl}_\gamma A$;
(3) $\mathrm{int}_\gamma(A) = X - \mathrm{cl}_\gamma(X-A)$;
(4) $\mathrm{cl}_\gamma(A) = X - \mathrm{int}_\gamma(X-A)$.

3. γ-continuous and γ-irresolute functions

Definition 3.1. Let (X,τ) and (Y,μ) be topological spaces. A function $f : X \to Y$ is called γ-continuous if the inverse image of each open set of Y is a γ-set in X.

Since the class of all γ-sets in a given topological space is also a topology, we get the following equivalent statements.

Theorem 3.2. *Let (X,τ) and (Y,μ) be topological spaces. If $f : (X,\tau) \to (Y,\mu)$ is a function, then the following statements are equivalent:*
(1) *f is γ-continuous;*
(2) *the inverse image of each closed set in Y is γ-closed;*
(3) $\mathrm{cl}_\gamma(f^{-1}(B)) \subset f^{-1}(\mathrm{cl}(B))$ *for every* $B \subset Y$;
(4) $f(\mathrm{cl}_\gamma(A)) \subset \mathrm{cl}(f(A))$ *for every* $A \subset X$;
(5) $f^{-1}(\mathrm{int}(B)) \subset \mathrm{int}_\gamma(f^{-1}(B))$ *for every* $B \subset Y$.

Theorem 3.3. *Let $f : (X,\tau) \to (Y,\mu)$ be a function between topological spaces. Then the following statements are equivalent:*
(1) *f is γ-continuous at x;*
(2) *if a filter F semi-converges to x, then $f(F)$ converges to $f(x)$;*
(3) *for $x \in X$ and for each neighborhood U of $f(x)$, there is a subset $V \in S_x$ such that $f(V) \subset U$.*

Proof. (1)⇒(2). Let V be any open neighborhood of $f(x)$ in Y. Then $f^{-1}(V)$ is a γ-set containing x. Thus $f^{-1}(V)$ is an element in S_x. Since F semi-converges to x and $f(F)$ is a filter, $V \in f(F)$. Consequently, $f(F)$ converges to $f(x)$.

(2)⇒(3). Let U be any γ-neighborhood of $f(x)$. Since always S_x semi-converges to x, from the hypothesis $S_{f(x)} \subset f(S_x)$, and so $U \in f(S_x)$. Thus, there is a subset $V \in S_x$ such that $f(V) \subset U$.

(3)⇒(1). The proof is obvious. □

We can easily verify the following result.

Corollary 3.4. *Let $f : (X,\tau) \to (Y,\mu)$ be a function. If f is semi-continuous at $x \in X$, then whenever a filter F semi-converges to x in X, $f(F)$ converges to $f(x)$ in Y.*

Remark 3.5. The following example shows that the converse of Corollary 3.4 may not be true. And we say that every γ-continuous function is semi-continuous.

Example 3.6. Let $\mathbb{R}$ be the set of real numbers with the usual topology. We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$, if $x \in Q$ and otherwise, $f(x) = \sqrt{2}$. Clearly, a filter F semi-converges to x if and only if $\dot{x} \subset F$. Thus $\dot{f}(x) \subset f(F)$ and so $f(F)$ converges to $f(x)$. For an open interval $(-1,1)$ containing 0, $f^{-1}\{(-1,1)\} = Q$. Since Q is not semi-open in $\mathbb{R}$, f is not semi-continuous.

Definition 3.7. Let (X,τ) and (Y,μ) be topological spaces. A function $f : X \to Y$ is called γ-irresolute if the inverse image of each γ set of Y is a γ-set in X.

The following theorems are obtained by Definition 3.7.

Theorem 3.8. *Let $f : (X,\tau) \to (Y,\mu)$ be a function between topological spaces. Then the following statements are equivalent:*
(1) *f is γ-irresolute;*
(2) *the inverse image of each γ-closed set in Y is a γ-closed set;*
(3) $\mathrm{cl}_{\gamma\tau}(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}_{\gamma\mu}(V))$ *for every $V \subset Y$;*
(4) $f(\mathrm{cl}_{\gamma\tau}(U)) \subset \mathrm{cl}_{\gamma\mu}(f(U))$ *for every $U \subset X$;*
(5) $f^{-1}(\mathrm{int}_{\gamma\mu}(B)) \subset \mathrm{int}_{\gamma\tau}(f^{-1}(B))$ *for every $B \subset Y$.*

Theorem 3.9. *Let $f : (X,\tau) \to (Y,\mu)$ be a function between topological spaces. Then the following statements are equivalent:*
(1) *f is γ-irresolute;*
(2) *for $x \in X$ and for each $V \in S_f(x)$, there exists an element U in the semi-neighborhood filter S_x such that $f(U) \subset V$;*
(3) *for each $x \in X$, if a filter F semi-converges to x, then $f(F)$ semi-converges to $f(x)$ in Y.*

Proof. (1)⇒(2). The proof is obvious.

(2)⇒(3). Let V be an element of the semi-neighborhood filter of $S_{f(x)}$ and U be an element of S_x and let F be a filter on X semi-converging to x. Then $f(S_x) \subset f(F)$. Since U is an element in S_x and $f(F)$ is a filter, we can say that $V \in f(F)$. Consequently, $f(F)$ semi-converges to $f(x)$.

(3)⇒(1). Let V be any γ-set in Y and suppose $f^{-1}(V)$ is not empty. For each $x \in f^{-1}(V)$, since the semi-neighborhood filter S_x semi-converges to x and the hypothesis, clearly, $f(S_x)$ semi-converges to x. And since V is γ-set containing $f(x)$ and $S_{f(x)} \subset f(S_x)$, $V \in f(S_x)$. Now we can take some γ-set U in S_x such that $f(U) \subset V$. Thus, $U \subset f^{-1}(V)$ and since S_x is a filter, so $f^{-1}(V)$ is an element of S_x. And $f^{-1}(V)$ is a γ-set in X from Theorem 2.7(b). □

Corollary 3.10. *Let $f : (X,\tau) \to (Y,\mu)$ be a function. If f is irresolute, then whenever a filter F semi-converges to x in X, $f(F)$ semi-converges to $f(x)$ in Y.*

Remark 3.11. We can get the following diagrams:

$$\begin{gathered}\text{continuity} \Longrightarrow \alpha\text{-continuity} \Longrightarrow \text{semi-continuity} \Longrightarrow \gamma\text{-continuity;}\\ \alpha\text{-irresolute} \Longrightarrow \text{irresolute} \Longrightarrow \gamma\text{-irresolute.}\end{gathered} \tag{3.1}$$

Definition 3.12. For two topological spaces (X,τ) and (Y,μ), a function $f : (X,\tau) \to (Y,\mu)$ is γ-open if for every open set G in X, $f(G)$ is a γ-set in Y.

Theorem 3.13. *Let $f : (X,\tau) \to (Y,\mu)$ be a function between topological spaces. Then, f is γ-open if and only if* $\mathrm{int}(f^{-1}(B)) \subset f^{-1}(\mathrm{int}_{\gamma\mu}(B))$, *for each $B \subset Y$.*

Proof. Let $B \subset Y$ and $x \in \operatorname{int}(f^{-1}(B))$. Then, $f(\operatorname{int}(f^{-1}(B)))$ is a γ-set containing $f(x)$. Since $f(\operatorname{int}(f^{-1}(B))) \in S_{f(x)}$ and $S_{f(x)}$ is a filter, $B \in S_{f(x)}$. Thus, $f(x) \in \operatorname{int}_{\gamma\mu}(B)$ and so $x \in f^{-1}(\operatorname{int}_{\gamma\mu}(B))$.

Conversely, let A be an open in X and $y \in f(A)$. Then,

$$A \subset \operatorname{int}(f^{-1}f(A)) \subset f^{-1}(\operatorname{int}_{\gamma\mu}(f(A))). \tag{3.2}$$

Let $x \in A$ be such that $f(x) = y$, then $x \in f^{-1}(\operatorname{int}_{\gamma\mu}(f(A)))$. Then, $y \in \operatorname{int}_{\gamma\mu}(f(A))$, and from Theorem 2.7(b) $f(A)$ is a γ-set. □

Remark 3.14. If any function is semi-open, then it is also γ-open. But the converse may not hold. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in \mathbb{R}$, where the real number set $\mathbb{R}$ with the usual topology. Then f is γ-open. For any semi-open set G, $f(G) = \{0\}$ and $\{0\}$ is not semi-open set, thus f is not semi-open.

Theorem 3.15. *Let $f : (X,\tau) \to (Y,\mu)$ be a function between topological spaces. The function f is γ-open if and only if for each $x \in X$ and for each neighborhood G of x, $f(G)$ is also an element of semi-neighborhood filter $S_{f(x)}$ in Y.*

Proof. Let G be a neighborhood of x, then there exists an open set U such that $x \in U \subset G$. Since f is γ-open, $f(x) \in f(U) = \operatorname{int}_{\gamma\mu}(f(U))$, and so $f(U) \in S_{f(x)}$. Since $S_{f(x)}$ is a filter, $f(G) \in S_{f(x)}$.

Conversely, let $B \subset Y$ and $x \in \operatorname{int}(f^{-1}(B))$, then since $\operatorname{int}(f^{-1}(B))$ is an element of S_x and S_x is a filter, $f^{-1}(B) \in S_x$. By the hypothesis $f(f^{-1}(B)) \in S_{f(x)}$, and since $S_{f(x)}$ is a filter, B is also an element of $S_{f(x)}$. By Definition 2.6, $f(x) \in \operatorname{int}_{\gamma\mu}(B)$ and by Theorem 3.13, the function f is γ-open. □

Remark 3.16. Now we get the following diagram:

$$\text{open function} \Longrightarrow \alpha\text{-open function} \Longrightarrow \text{semi-open function} \Longrightarrow \gamma\text{-open function}. \tag{3.3}$$

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