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BOUNDED SOLUTIONS FOR FUZZY INTEGRAL EQUATIONS

D. N. GEORGIOU and I. E. KOUGIAS

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We study conditions under which the solutions of a fuzzy integral equation are bounded.

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1. Introduction. The concept of set-valued functions and their calculus [2] were found useful in some problems in economics [3], as well as in control theory [17]. Later on, the notion of H-differentiability was introduced by Puri and Ralesku in order to extend the differential of set-valued functions to that of fuzzy functions [29]. This in turn led Seikkala [30] to introduce the notion of fuzzy derivative, which is a generalization of the Hukuhara derivative and the fuzzy integral, which is the same as that proposed by Dubois and Prade [7, 8]. A natural consequence of the above was the study of fuzzy differential and integral equations, see [5, 9, 10, 11, 12, 18, 19, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33].

Fixed point theorems for fuzzy mappings, an important tool for showing existence and uniqueness of solutions to fuzzy differential and integral equations, have recently been proved by various authors, see [1, 4, 13, 14, 15, 16, 22, 27]. In particular, in [22] Lakshmikantham and Vatsala proved the existence of fixed points to fuzzy mappings, using theory of fuzzy differential equations. Finally, stability criteria for the solutions of fuzzy differential systems are given in [21].

In this paper, we examine conditions under which all the solutions of the fuzzy integral equation

$$x(t) = \int_0^t G(t,s)x(s)\,ds + f(t) \tag{1.1}$$

and the special case

$$x(t) = \int_0^t k(t-s)x(s)\,ds + f(t) \tag{1.2}$$

are bounded.

These fuzzy integral equations are proved useful when studying observability of fuzzy dynamical control systems, see [6].

2. Preliminaries. By $P_k(\mathbb{R}^n)$, we denote the family of all nonempty compact convex subsets of $\mathbb{R}^n$.

For $A, B \in P_k(\mathbb{R}^n)$, the Hausdorff metric is defined by

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\ \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}. \tag{2.1}$$

A *fuzzy set* in $\mathbb{R}^n$ is a function with domain $\mathbb{R}^n$ and values in $[0,1]$, that is an element of $[0,1]^{\mathbb{R}^n}$ (see [35, 34]).

Let $u \in [0,1]^{\mathbb{R}^n}$, the *a-level set* is

$$\begin{aligned} [u]^a &= \{x \in \mathbb{R}^n : u(x) \geq a\}, \quad a \in (0,1], \\ [u]^0 &= \mathrm{Cl}\left(\{x \in \mathbb{R}^n : u(x) > 0\}\right). \end{aligned} \tag{2.2}$$

By E^n, we denote the family of all fuzzy sets $u \in [0,1]^{\mathbb{R}^n}$ (see [18, 29, 35]), for which:

(i) u is normal, that is, there exists an element $x_0 \in \mathbb{R}^n$, such that $u(x_0) = 1$,

(ii) u is fuzzy convex,

(iii) u is uppersemicontinuous,

(iv) $[u]^0$ is compact.

Let $u \in E^n$. Then for each $a \in (0,1]$ the a-level set $[u]^a$ of u is a nonempty compact convex subset of $\mathbb{R}^n$, that is, $u \in P_k(\mathbb{R}^n)$. Also $[u]^0 \in P_k(\mathbb{R}^n)$.

Let

$$D : E^n \times E^n \longrightarrow [0,\infty), \qquad D(u,v) = \sup\{d([u]^a,[v]^a) : a \in [0,1]\}, \tag{2.3}$$

where d is the Hausdorff metric for nonempty compact convex subsets of $\mathbb{R}^n$ (see [18]).

3. Main results

Notation 3.1. By $\hat{0} \in E^n$, we denote the fuzzy set for which $\hat{0}(x) = 1$ if $x = 0$ and $\hat{0}(x) = 0$ if $x \neq 0$.

Definition 3.2. A mapping $x : T \to E^n$ is bounded, where T is an interval of the real line, if there exists an element $r > 0$, such that

$$D(x(t),\hat{0}) < r, \quad \forall t \in T. \tag{3.1}$$

Theorem 3.3. *Suppose that $f : [0,+\infty) \to E^n$ with $D(f(t),\hat{0}) \leq M$, and $G : \Delta \to \mathbb{R}$ is continuous, where $\Delta = \{(t,s) : 0 \leq s \leq t < \infty\}$. If there exists $m < 1$ with $\int_0^t |G(t,s)|\,ds \leq m$, for $t \in [0,+\infty)$, then all solutions of the fuzzy integral equation*

$$x(t) = \int_0^t G(t,s)x(s)\,ds + f(t), \tag{3.2}$$

are bounded.

Proof. Let $x(t)$ be an unbounded solution of (3.2). Then for every $r > 0$, there exists an element $t_1 \in (0,\infty)$, such that

$$D(x(s),\hat{0}) < r, \quad \forall s \in [0,t_1), \qquad D(x(t_1),\hat{0}) = r. \tag{3.3}$$

Clearly, we can find a positive number r with

$$M + mr < r. \tag{3.4}$$

By (3.3), (3.4), and the assumptions of the theorem we have

$$\begin{aligned} r &= D(x(t_1),\hat{0}) \\ &= D\left(\int_0^{t_1} G(t_1,s)x(s)\,ds + f(t_1),\hat{0}\right) \\ &\le D\left(\int_0^{t_1} G(t_1,s)x(s)\,ds,\hat{0}\right) + D(f(t_1),\hat{0}) \\ &\le \int_0^{t_1} D(G(t_1,s)x(s),\hat{0})\,ds + M \quad \text{(see [19, Theorem 4.3])} \\ &\le \int_0^{t_1} |G(t_1,s)|D(x(s),\hat{0})\,ds + M \quad \text{(by the definition of } D\text{, see [19])} \\ &\le M + mr < r, \end{aligned} \tag{3.5}$$

which is a contradiction. Thus $x(t)$ is bounded. □

Theorem 3.4. *Suppose that $f : [0,\infty) \to E^n$ and $k : [0,\infty) \to \mathbb{R}$ are continuous and that there exist constants A, B, and $a > 0$ with $0 < B < 1$ and*

$$D(f(t),\hat{0}) \le Ae^{-at}, \qquad \int_0^t |k(t-s)|\,ds \le Be^{-at}. \tag{3.6}$$

Then, every solution of the fuzzy integral equation

$$x(t) = \int_0^t k(t-s)x(s)\,ds + f(t) \tag{3.7}$$

is bounded.

Proof. Let $x(t)$ be an unbounded solution of (3.7). Then for every $r > 0$, there exists an element $t_1 \in (0,\infty)$, such that

$$D(x(s),\hat{0}) < r, \quad \forall s \in [0,t_1), \qquad D(x(t_1),\hat{0}) = r. \tag{3.8}$$

Clearly, we can find a positive number r with

$$A+Br<r. \tag{3.9}$$

By (3.6), (3.7), and (3.8) we have

$$\begin{aligned}
r &= D(x(t_1),\hat{0}) \\
&= D\left(\int_0^{t_1} k(t_1-s)x(s)\,ds + f(t_1),\hat{0}\right) \\
&\le D\left(\int_0^{t_1} k(t_1-s)x(s)\,ds,\hat{0}\right) + D(f(t_1),\hat{0}) \\
&\le \int_0^{t_1} D(k(t_1-s)x(s),\hat{0})\,ds + D(f(t_1),\hat{0}) \quad \text{(see [19, Theorem 4.3])} \\
&\le \int_0^{t_1} |k(t_1-s)|D(x(s),\hat{0})\,ds + D(f(t_1),\hat{0}) \quad \text{(by the definition of } D\text{, see [19])} \\
&\le Ae^{-at_1} + Be^{-at_1}r \\
&\le Ae^{-at_1} + Be^{-at_1}r
\end{aligned} \tag{3.10}$$

and thus

$$e^{at_1}r < A+Br<r, \tag{3.11}$$

which is a contradiction. Thus, $x(t)$ is bounded. □

Remark 3.5. Now, since the initial value problem

$$x'(t)=f(t,x(t)), \quad t\in T,\ x(0)=x_0, \tag{3.12}$$

where $f:T\times E^n\to E^n$ is continuous, it is equivalent to the integral equation

$$x(t)=x_0+\int_0^t f(s,x(s))\,ds, \quad t\in[0,b] \quad \text{(see [20, Lemma 3.1])}. \tag{3.13}$$

If for the map $f:T\times E^n\to E^n$ the conditions of Theorem 3.3 or 3.4 hold true, then all the solutions of the initial value problem (3.12) are bounded.

4. Conclusion. In this paper, using a Gronwall type inequality, we give conditions under which the fuzzy integral equations (3.2) and (3.7) possess only bounded solutions. Consequently, this implies that the Cauchy problem (3.12) possesses only bounded solutions as well. It appears that, these fuzzy equations are useful when one studies the observability of fuzzy dynamical control systems. We also think that, our results can be of use in studying stability of fuzzy differential equations and fuzzy differential systems.

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D. N. Georgiou: University of Patras, Faculty of Sciences, Department of Mathematics, 265 00 Patras, Greece
E-mail address: georgiou@math.upatras.gr

I. E. Kougias: Technological Educational Institute of Epirus, School of Business Administration and Economics, Department of Accounting, 481 00 Preveza, Greece
E-mail address: kougias@teiep.gr