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$A_r(\lambda)$-WEIGHTED CACCIOPPOLI-TYPE AND POINCARÉ-TYPE INEQUALITIES FOR A-HARMONIC TENSORS

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We prove a local version of weighted Caccioppoli-type inequality, then we prove a version of weighted Poincaré-type inequality for A-harmonic tensors both locally and globally.

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1. Introduction. There have been many studies for the integrability of differential forms, and the estimations of the integrals of differential forms. As extensions of those studies, the integrability and estimations of integrals for A-harmonic tensors are also studied and applied in many fields such as in tensor analysis, potential theory, partial differential equations, and quasiregular mappings, see [1, 2, 6, 7, 8, 9, 10, 11, 12]. There are many studies about Caccioppoli-type and Poincaré-type inequalities for A-harmonic tensors, see [3, 4, 5, 12]. We state the following specific results from [12].

Theorem 1.1. *Let u be an A-harmonic tensor in Ω and let $\sigma > 1$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B} \le C|B|^{-1}\|u-c\|_{s,\sigma B} \tag{1.1}$$

for all balls or cubes B with $\sigma B \subset \Omega$.

Theorem 1.2. *Let $u \in D'(Q,\wedge^l)$ and $du \in L^p(Q,\wedge^{l+1})$. Then there exists a closed form u_Q, defined in Q, such that $u-u_Q$ is in $W_p^1(Q,\wedge^l)$ with $1 < p < \infty$ and*

$$||u-u_Q||_{p,Q} \le C(n,p)|Q|^{1/n}\|du\|_{p,Q} \tag{1.2}$$

for Q a cube or a ball in $\mathbb{R}^n$, $l = 0,1,\dots,n$.

Our work is to give new versions of Theorems 1.1 and 1.2 with $A_r(\lambda)$ weight. When $\lambda = 1$, the properties of $A_r(1)$-weight can be found in [6].

We first introduce some related definitions and notations which are adopted from [12].

We assume that Ω is a connected open subset of $\mathbb{R}^n$. The Lebesgue measure of a set $E \subset \mathbb{R}^n$ is denoted by $|E|$. Balls in $\mathbb{R}^n$ are denoted by B and σB is the ball with the same center as B and with $\mathrm{diam}(\sigma B) = \sigma\,\mathrm{diam}(B)$. We call w a weight if $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $w > 0$ a.e.

Let $e_1, e_2, \dots, e_n$ be the standard unit basis of $\mathbb{R}^n$. Assume that $\wedge^l = \wedge^l(\mathbb{R}^n)$ is the linear space of l-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$ corresponding to all ordered l-tuples $I = (i_1, i_2, \dots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. The Hodge star operator $* : \wedge \to \wedge$ is defined by $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by $|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \wedge^0 = \mathbb{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $* : \wedge^l \to \wedge^{n-l}$ and $** (-1)^{l(n-l)} : \wedge^l \to \wedge^l$.

For $0 \le k \le n$, a k-form $\omega(x) \in \wedge^k(\mathbb{R})^n$ is defined by

$$\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1, i_2, \dots, i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}, \tag{1.3}$$

where $\omega_{i_1, i_2, \dots, i_k}(x)$ are real functions in $\mathbb{R}^n$, $I = (i_1, i_2, \dots, i_k)$, $i_j \in \{1, 2, \dots, n\}$ and $j = 1, 2, \dots, k$. The function $\omega(x)$ is called a differential k-form if $\omega_{i_1, i_2, \dots, i_k}(x)$ are differentiable functions. Note that a differential 0-form is a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. A differential l-form ω on Ω is a locally integrable function or more generally, a Schwartz distribution on Ω with values in $\wedge^l(\mathbb{R}^n)$. We denote $D'(\Omega, \wedge^l)$ as a space of all differential l-forms and $L^p(\Omega, \wedge^l)$ as a space of differential l-forms with coefficients in the $L^p(\Omega, \mathbb{R}^n)$. The space $L^p(\Omega, \wedge^l)$ is a Banach space with the norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}. \tag{1.4}$$

We also denote $W_p^1(\Omega, \wedge^l)$ as a space of differential l-forms on Ω whose coefficients are in Sobolev space $W_p^1(\Omega, \mathbb{R})$.

An A-harmonic equation for differential forms is

$$d^\star A(x, d\omega) = 0, \tag{1.5}$$

where $d^\star : D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$, as the formal adjoint operator of d, is given by

$$d^\star = (-1)^{nl+1} \star d \star \tag{1.6}$$

on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \dots, n$, and $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ satisfies the following conditions:

$$|A(x, \xi)| \le a |\xi|^{p-1}, \qquad \langle A(x, \xi), \xi \rangle \ge |\xi|^p \tag{1.7}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.5). Let $W_{p,\mathrm{loc}}^1(\Omega, \wedge^{l-1}) = \cap W_p^1(\Omega', \wedge^{l-1})$, where the intersection is for all Ω' compactly contained in Ω. A solution to (1.5) is an element of the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0 \tag{1.8}$$

for all $\varphi \in W^1_p(\Omega, \wedge^{l-1})$ with compact support, see [8, 9, 12].

Definition 1.3. We call u an A-harmonic tensor in Ω if u satisfies the A-harmonic equation (1.5) in Ω.

The following definition belongs to Ding and Shi [5].

Definition 1.4. The weight $w(x)$ satisfies the $A_r(\lambda)$ condition, $r > 1$, $\lambda > 1$, write $w \in A_r(\lambda)$, if $w(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|}\int_B w^\lambda dx\right)\left(\frac{1}{|B|}\int_B \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)} < \infty \tag{1.9}$$

for any ball $B \subset \mathbb{R}^n$.

The following lemma is from Nolder [12].

Lemma 1.5. *Each Ω has a modified Whitney cover of cubes $W = \{Q_i\}$ which satisfy*

$$\cup Q_i = \Omega, \qquad \sum_{Q\in W} \chi_{\sqrt{5/4}Q} \le N\chi_\Omega \tag{1.10}$$

for all $x \in \mathbb{R}^n$ and some $N > 1$ and if $Q_i \cap Q_j \ne \emptyset$, then there exists a cube R $(\notin W)$ in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$.

We also need the following generalized Hölder's inequality.

Lemma 1.6. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If f and g are measurable functions on $\mathbb{R}^n$, then*

$$\|fg\|_{s,\Omega} \le \|f\|_{\alpha,\Omega}\|g\|_{\beta,\Omega} \tag{1.11}$$

for any $\Omega \subset \mathbb{R}^n$.

2. Main results

Theorem 2.1. *Let $u \in D'(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$, be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the A-harmonic equation and weight $w \in A_r(\lambda)$ for some $r > 1$ and $\lambda > 0$. Then there exists a constant C, independent of u and du, such that*

$$\left(\int_B |du|^s w^{s\lambda/(1+s)}\,dx\right)^{1/s} \le \frac{C}{|B|}\left(\int_{\rho B} |u-c|^s w^{s/(1+s)}\,dx\right)^{1/s} \tag{2.1}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

Proof. Choose $t = s(s+1)$ for given $1 < s < \infty$, then $1 < s < t$. Applying Hölder's inequality (1.11) and Theorem 1.1, we obtain

$$\begin{aligned}\left(\int_B |du|^s w^{s\lambda/(s+1)}\,dx\right)^{1/s} &= \left(\int_B \left(|du| w^{\lambda/(s+1)}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |du|^t\,dx\right)^{1/t}\left(\int_B \left(w^{\lambda/(s+1)}\right)^{st/(t-s)}dx\right)^{(t-s)/(st)} \\ &\le C_1|B|^{-1}\|u-c\|_{t,\rho B}\left(\int_B w^{\lambda}\,dx\right)^{1/(s+1)} \\ &\le C_1|B|^{-1}\left(\int_B w^{\lambda}\,dx\right)^{1/(s+1)}|\rho B|^{(m-t)/mt}\|u-c\|_{m,\rho B},\end{aligned} \tag{2.2}$$

where the last inequality was obtained by choosing $m = (s+s^2)/(sr+1)$ and applying (1.11) to $\|u-c\|_{t,\rho B}$ with $1/t = 1/m + (m-t)/mt$.

Since $m > s$, using Hölder's inequality, we have

$$\begin{aligned}&\left(\int_{\rho B} |u-c|^m\,dx\right)^{1/m} \\ &\quad= \left(\int_{\rho B}\left(|u-c| w^{1/(s+1)} w^{-1/(s+1)}\right)^m dx\right)^{1/m} \\ &\quad\le \left(\int_{\rho B}\left(|u-c| w^{1/(s+1)}\right)^s dx\right)^{1/s}\left(\int_{\rho B}\left(w^{-1/(s+1)}\right)^{ms/(s-m)}dx\right)^{(s-m)/ms}.\end{aligned} \tag{2.3}$$

By the choice of m, $(s-m)(s+1)/ms = r-1$. Since $w \in A_r(\lambda)$, we have

$$\begin{aligned}&\left(\int_B w^{\lambda}\,dx\right)^{1/(s+1)}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{ms/(s-m)(s+1)}dx\right)^{(s-m)(s+1)/(ms(s+1))} \\ &\quad= \left(\left(\int_B w^{\lambda}\,dx\right)\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right)^{1/(s+1)} \\ &\quad\le \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w^{\lambda}\,dx\right)\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right)^{1/(s+1)} \\ &\quad\le C_2|\rho B|^{r/(s+1)}.\end{aligned} \tag{2.4}$$

Thus, putting (2.2), (2.3), and (2.4) together and noting that $(m-t)/mt = -r/(s+1)$, we have

$$\begin{aligned}
&\left(\int_B |du|^s w^{s\lambda/(s+1)}\,dx\right)^{1/s}\\
&\quad\le C_1|B|^{-1}|\rho B|^{(m-t)/mt}C_2|\rho B|^{r/(s+1)}\left(\int_{\rho B}|u-c|^s w^{s/(s+1)}\,dx\right)^{1/s}\\
&\quad\le C_3|B|^{-1}\left(\int_{\rho B}|u-c|^s w^{s/(s+1)}\,dx\right)^{1/s}.
\end{aligned}\tag{2.5}$$

We have proved Theorem 2.1. □

By choosing different values of λ in Theorem 2.1, we get different versions of the Caccioppoli-type inequality. For example, if $\lambda = (s+1)/s$, (2.1) reduces to

$$\left(\int_B |du|^s w\,dx\right)^{1/s} \le \frac{C}{|B|}\left(\int_{\rho B}|u-c|^s w^{s/(s+1)}\,dx\right)^{1/s}. \tag{2.6}$$

If we choose $\lambda = s+1$ in Theorem 2.1, (2.1) is in the form of

$$\left(\int_B |du|^s w^s\,dx\right)^{1/s} \le \frac{C}{|B|}\left(\int_{\rho B}|u-c|^s w^{s/(s+1)}\,dx\right)^{1/s}. \tag{2.7}$$

If $\lambda = 1$, we get the symmetric form of (2.1),

$$\left(\int_B |du|^s w^{s/(s+1)}\,dx\right)^{1/s} \le \frac{C}{|B|}\left(\int_{\rho B}|u-c|^s w^{s/(s+1)}\,dx\right)^{1/s}. \tag{2.8}$$

Now we consider a type of Poincaré inequality with $A_r(\lambda)$-weights. The following is a local result for any connected open subset in $\mathbb{R}^n$.

Theorem 2.2. *Let $u \in D'(\Omega, \wedge^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $du \in L^s(\Omega, \wedge^{l+1})$, $l = 0, 1, \dots, n$. Assume that $\sigma > 1$, $1 < s < \infty$, and $w \in A_r(\lambda)$ for some $r > 1$ and any real number $\lambda > 0$. Then*

$$\left(\frac{1}{|B|}\int_B |u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} \le C|B|^{1/n}\left(\frac{1}{|B|}\int_{\sigma B}|du|^s w^{1/s}\,dx\right)^{1/s} \tag{2.9}$$

for all balls B with $\sigma B \subset \Omega$. Here C is a constant independent of u.

Proof. We only need to prove the following:

$$\left(\int_B |u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} \le C|B|^{1/n}\left(\int_{\sigma B}|du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.10}$$

For any $1 < s < \infty$, choose t such that $t = s^2/(s-1)$, then $1 < s < t$. By Hölder's inequality we have

$$\begin{aligned}\left(\int_B |u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} &= \left(\int_B \left(|u-u_B|w^{\lambda/s^2}\right)^s dx\right)^{1/s} \\ &\le \left(\int_B |u-u_B|^t\,dx\right)^{1/t}\left(\int_B \left(w^{\lambda/s^2}\right)^{st/(t-s)} dx\right)^{(t-s)/st} \\ &= \left(\int_B |u-u_B|^t\,dx\right)^{1/t}\left(\int_B w^{\lambda}\,dx\right)^{1/s^2}.\end{aligned} \tag{2.11}$$

Using Hölder's inequality again and choosing m such that $m = s^2/(s+r-1)$ for any given $r > 1$, we obtain

$$\left(\int_B |u-u_B|^t\,dx\right)^{1/t} \le |\sigma B|^{(m-t)/(mt)}\left(\int_{\sigma B} |u-u_B|^m\,dx\right)^{1/m} \tag{2.12}$$

for any $\sigma > 1$. Thus, by Theorem 1.2,

$$\left(\int_B |u-u_B|^t\,dx\right)^{1/t} \le |\sigma B|^{(m-t)/mt} C(n,m)|\sigma B|^{1/n}\left(\int_{\sigma B} |du|^m\,dx\right)^{1/m}, \tag{2.13}$$

where

$$\begin{aligned}\left(\int_{\sigma B} |du|^m\,dx\right)^{1/m} &= \left(\int_{\sigma B} |du|^m w^{1/s^2} w^{-1/s^2}\,dx\right)^{1/m} \\ &\le \left(\int_{\sigma B} \left(|du|w^{1/s^2}\right)^s dx\right)^{1/s}\left(\int_{\sigma B} \left(w^{-1/s^2}\right)^{ms/(s-m)} dx\right)^{(s-m)/ms} \\ &= \left(\int_{\sigma B} |du|^s w^{1/s}\,dx\right)^{1/s}\left(\int_{\sigma B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/s^2}.\end{aligned} \tag{2.14}$$

Substituting (2.12), (2.13), and (2.14) to inequality (2.11), and using $A_r(\lambda)$ condition to w, we have

$$\begin{aligned}&\left(\int_B |u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} \\ &\quad\le C(n,m)|\sigma B|^{1/n}\left(\int_B w^{\lambda}\,dx\right)^{1/s^2} |\sigma B|^{(m-t)/mt}\left(\int_{\sigma B} |du|^s w^{1/s}\,dx\right)^{1/s} \\ &\qquad\times\left(\int_{\sigma B} \left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/s^2}\end{aligned}$$

$$
\begin{aligned}
&\le C(n,m)|\sigma B|^{1/n}\left(\int_{\sigma B}|du|^s w^{1/s}\,dx\right)^{1/s}\\
&\quad\times\left[\left(\frac{1}{|\sigma B|}\int_{\sigma B}w^{\lambda}\,dx\right)\left(\frac{1}{|\sigma B|}\int_{\sigma B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right]^{1/s^2}\\
&\le C_1(n,m)|B|^{1/n}\left(\int_{\sigma B}|du|^s w^{1/s}\,dx\right)^{1/s}.
\end{aligned}
\tag{2.15}
$$

□

Theorem 2.3 (global result of Theorem 2.2). *Let $u \in D'(\Omega,\wedge^l)$ be an A-harmonic tensor in a domain $\Omega \subset \mathbb{R}^n$ and $du \in L^s(\Omega,\wedge^{l+1})$, $l = 0,1,\dots,n$. Assume that $1 < s < \infty$ and $w \in A_r(\lambda)$ for some $r > 1$ and $\lambda > 0$. Then*

$$
\left(\frac{1}{|\Omega|}\int_{\Omega}|u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} \le C|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_{\Omega}|du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.16}
$$

Proof. By Lemma 1.5, there exists a Whitney cover $F = \{Q_i\}$ of Ω. In particular, we can choose $1 < \sigma \le \sqrt{5/4}$ in Theorem 2.2, so that

$$
\begin{aligned}
\int_{\Omega}|u-u_B|^s w^{\lambda/s}\,dx &\le \int_{\sum_{Q\in F}Q}|u-u_B|^s w^{\lambda/s}\,dx\\
&\le \sum_{Q\in F}\int_{Q}|u-u_B|^s w^{\lambda/s}\,dx\\
&\le \sum_{Q\in F}C_1|Q|^{s/n}\int_{\sigma Q}|du|^s w^{1/s}\,dx\\
&\le C_2|\Omega|^{s/n}\sum_{Q\in F}\int_{\sigma Q}|du|^s w^{1/s}\,dx\chi_{\sigma Q}\\
&\le C_3|\Omega|^{s/n}\sum_{Q\in F}\int_{\Omega}|du|^s w^{1/s}\,dx\chi_{\sqrt{5/4}Q}\\
&\le C_4|\Omega|^{s/n}\int_{\Omega}|du|^s w^{1/s}\,dx\sum_{Q\in F}\chi_{\sqrt{5/4}Q}\\
&\le C_5|\Omega|^{s/n}\int_{\Omega}|du|^s w^{1/s}\,dxN\chi_{\Omega}(x)\\
&\le C_6|\Omega|^{s/n}\int_{\Omega}|du|^s w^{1/s}\,dx.
\end{aligned}
\tag{2.17}
$$

Thus,

$$
\frac{1}{|\Omega|}\int_{\Omega}|u-u_B|^s w^{\lambda/s}\,dx \le C_6\frac{1}{|\Omega|}|\Omega|^{s/n}\int_{\Omega}|du|^s w^{1/s}\,dx \tag{2.18}
$$

or

$$
\left(\frac{1}{|\Omega|}\int_{\Omega}|u-u_B|^s w^{\lambda/s}\,dx\right)^{1/s} \le C_6|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_{\Omega}|du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.19}
$$

□

Remark 2.4. Similar to Theorem 2.1, we have different versions of global results for Poincaré-type inequality by choosing different values of λ. For instance, as $\lambda = 1$, (2.16) reduces to

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_B|^s w^{1/s}\,dx\right)^{1/s} \le C|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_\Omega |du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.20}$$

As $\lambda = s$, (2.16) is in the form of

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_B|^s w\,dx\right)^{1/s} \le C|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_\Omega |du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.21}$$

And if $\lambda = s^2$ in (2.16), we have

$$\left(\frac{1}{|\Omega|}\int_\Omega |u-u_B|^s w^s\,dx\right)^{1/s} \le C|\Omega|^{1/n}\left(\frac{1}{|\Omega|}\int_\Omega |du|^s w^{1/s}\,dx\right)^{1/s}. \tag{2.22}$$

Since parameter $\lambda > 0$ can be chosen arbitrarily, the inequalities in our theorems can be used to estimate a relatively broad class of integrals.

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