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OPTIMAL BOUND FOR THE NUMBER OF (-1)-CURVES ON EXTREMAL RATIONAL SURFACES

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We give an optimal bound for the number of (-1)-curves on an extremal rational surface X under the assumption that $-K_X$ is numerically effective and having self-intersection zero. We also prove that a nonelliptic extremal rational surface has at most nine (-1)-curves.

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1. Introduction. Let X be a smooth projective rational surface defined over the field of complex numbers. From now on we assume that $-K_X$ is numerically effective (in short NEF, i.e., the intersection number of the divisor K_X with any effective divisor on X is less than or equal to zero, where K_X is a canonical divisor on X) and of self-intersection zero.

It is easy to see that X is obtained by blowing up 9 points (possibly infinitely near) of the projective plane.

Nagata [4] proved that if the 9 points are in general positions, then X has an infinite number of (-1)-curves (i.e., smooth rational curves of self-intersection -1).

Miranda and Persson [3] studied the case when the position of the 9 points give a rational elliptic surface with a section. They classified all such surfaces which have a finite number of (-1)-curves and called them extremal Jacobian elliptic rational surfaces. For each case, they gave the number of (-1)-curves.

We use the following notations:

(i) $\sim$ is the linear equivalence of divisors on X;

(ii) $[D]$ is the set of divisors D' on X such that $D' \sim D$;

(iii) $\mathrm{Div}(X)$ is the group of divisors on X;

(iv) $NS(X)$ is the quotient group $\mathrm{Div}(X)/\sim$ of $\mathrm{Div}(X)$ by $\sim$ (the linear, algebraic, and numerical equivalences are the same on $\mathrm{Div}(X)$ since X is a rational surface);

(v) $D \cdot D'$ denotes the intersection number of the divisor D with the divisor D', in particular the self-intersection of D is $D^2 = D \cdot D$;

(vi) $\overline{D}$ is the element associated to D in $NS(X) \otimes \mathbb{Q}$.

Following [3], we define a smooth rational projective surface having a finite number of (-1)-curves on it as an extremal rational surface. The extremal rational surfaces are classified by the following theorem which can be found in [1, Theorem 3.1, page 65].

Theorem 1.1. *Let X be a smooth projective rational surface having $-K_X$ NEF and of self-intersection zero. Then the following statements are equivalent:*

(1) *X is extremal;*
(2) *X satisfies the following two conditions:*
  (a) *the rank of the matrix $(C_i \cdot C_j)_{i,j=1,\dots,r}$ is equal to 8, where $\{C_i : i = 1,\dots,r\}$ is the finite set of (-2)-curves on X; a (-2)-curve is a smooth rational curve of self-intersection -2;*
  (b) *there exist r strictly positive rational numbers a_i, $i = 1,\dots,r$, such that $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$.*

From this theorem we deduce the following lemma.

Lemma 1.2. *Let X be an extremal surface. With the same notation as Theorem 1.1, if all of the a_i, $i = 1,\dots,r$, are strictly positive integers, then a (-1)-curve on X meets only one (-2)-curve C_i in one point and necessarily the coefficient a_i of C_i must be equal to one.*

Proof. Let E be a (-1)-curve on X. We have $\sum_{i=1}^{i=r} a_i E \cdot C_i = 1$ (since $-\overline{K}_X = \sum_{i=1}^{i=r} a_i \overline{C}_i$ and E is a (-1)-curve). On the other hand, for every $j \in \{1,2,\dots,r\}$, the intersection number of E with C_j is a nonnegative integer. Therefore, there exists $i \in \{1,2,\dots,r\}$ such that $a_i E \cdot C_i = 1$ and for every $j \in \{1,2,\dots,r\}$, $j \neq i$, $E \cdot C_j = 0$. Hence the lemma follows. □

In this note, we give an optimal bound for the number of (-1)-curves on an extremal rational surface. Keeping the same notations as in Theorem 1.1, our result is as follows.

Theorem 1.3. *Let X be an extremal rational surface. The number of (-1)-curves on X is bounded by the integer*

$$-1 + \prod_{i=1}^{i=r} \left(1 + \left[\left[\frac{1}{a_i}\right]\right]\right), \tag{1.1}$$

where $[[\quad]]$ denotes the greatest integer function. This bound is optimal.

2. The proof.

Let X be a smooth projective rational surface such that $K_X^2 = 0$, where K_X is a canonical divisor of X. We assume that $-K_X$ is NEF, that is, $K_X \cdot D \le 0$ for every effective divisor D on X.

For each $(r+2)$-tuple $(p,q;n_1,\dots,n_r)$ of integers, where r is a strictly positive integer, we consider the set $\mathscr{E}_{p,q}^{n_1,\dots,n_r}$ of divisor classes $[D]$ on X such that

(i) $D^2 = p$,
(ii) $D \cdot K_X = q$,
(iii) $D \cdot C_i = n_i$ for each $i = 1,\dots,r$, where $\{C_i : i = 1,\dots,r\}$ is the finite set of (-2)-curves on X.

We think of $\mathscr{E}_{p,q}^{n_1,\dots,n_r}$ as a set of elements of $NS(X)$ with imposed intersection with the set of (-2)-curves like a linear system with imposed base points. We prove that if the set of (-2)-curves on X is maximal in a sense that will be explained in Proposition 2.1, then for each nonzero integer q, the set $\mathscr{E}_{p,q}^{n_1,\dots,n_r}$ has at most one element.

Proposition 2.1. *Let X be a smooth projective rational surface having an anticanonical divisor $-K_X$ of self-intersection zero. If the set of (-2)-curves on X spans the orthogonal complement of K_X, then for each $(r+2)$-tuple $(p,q;n_1,\dots,n_r)$ of integers, with q nonzero, the set $\mathscr{E}_{p,q}^{n_1,\dots,n_r}$ has at most one element.*

Proof. If the set $\mathscr{E}_{p,q}^{n_1,\dots,n_r}$ is not empty, consider two elements $[D]$ and $[D']$. First, we have $D-D'$ belongs to the orthogonal complement of K_X since $D\cdot K_X = q = D'\cdot K_X$, keeping in mind that $D-D'$ is orthogonal to each C_i, for $i=1,\dots,r$, (since $D\cdot C_i = D'\cdot C_i$ for each $i = i = 1,\dots,r$) and the fact that the set of (-2)-curves on X spans the orthogonal complement of K_X, we conclude that $(D-D')^2 = 0$. Hence there exists a rational number m such that $\overline{D} = \overline{D'} + m\overline{K_X}$. Furthermore $D^2 = D'^2$. Since $q \neq 0$, we have $m = 0$ and hence D is linearly equivalent to D', that is, $[D] = [D']$. □

An immediate consequence is the following corollary.

Corollary 2.2. *Let X be a smooth projective rational surface having an anticanonical divisor $-K_X$ of self-intersection zero. If the set of (-2)-curves on X spans the orthogonal complement of K_X, then for two different (-1)-curves E and E' on X, there exists $i\in\{1,\dots,r\}$ such that $C_i\cdot E \neq C_i\cdot E'$, where $\{C_1,\dots,C_r\}$ is the set of (-2)-curves on X.*

Proof of Theorem 1.3. Let E be a (-1)-curve on X. From Theorem 1.1(2)(b), we have $0 \le E\cdot C_i \le [[1/a_i]]$ for each $i=1,\dots,r$. The fact that $E\cdot(-K_X) = 1$ implies that there exists $j_E \in \{1,\dots,r\}$ such that $E\cdot C_{j_E} \ge 1$, so the r-tuple $(E\cdot C_i)_{i=1,\dots,r}$ of integers belongs to the set $\prod_{i=1}^{i=r}([0,[[1/a_i]]]\cap\mathbb{N})\setminus\{(0,\dots,0)\}$ which has exactly $-1+\prod_{i=1}^{i=r}(1+[[1/a_i]])$ elements. Consider the map ϕ defined from the set of (-1)-curves on X to $\prod_{i=1}^{i=r}([0,[[1/a_i]]]\cap\mathbb{N})\setminus\{(0,\dots,0)\}$, it is given by $\phi(E) = (E\cdot C_i)_{i=1,\dots,r}$ for every (-1)-curve E on X. Corollary 2.2 confirm that ϕ is injective. Therefore, the first result of Theorem 1.3 holds.

The suggested bound is optimal for certain extremal rational surfaces (see Remark 2.3). □

Remark 2.3. It is interesting to know that for which extremal rational surfaces, the set of (-1)-curves is in one-to-one correspondence with $\prod_{i=1}^{i=r}([0,[[1/a_i]]]\cap\mathbb{N})\setminus\{(0,\dots,0)\}$. For example, in the case of an extremal elliptic Jacobian rational surface [3, Table 5.1, page 544], the only such surfaces for which there is a bijection are

(i) the surface X_{22} which has the set $\{II, II^*\}$ as set of singular fibers;

(ii) the surface X_{211} which has the set $\{II^*, I_1, I_1\}$ as set of singular fibers.

More generally, for a given extremal surface X, we ask: which r-tuple $(n_1,\dots,n_r)$ of $\prod_{i=1}^{i=r}([0,[[1/a_i]]]\cap\mathbb{N})\setminus\{(0,\dots,0)\}$ represent a (-1)-curve? Very little is known about this question.

Remark 2.4. Let X be an extremal rational surface which is not elliptic, then we have the following facts:

(1) the set of (-2)-curves on X is connected and hence has one of the three types of configurations $\tilde{A}_8$, $\tilde{D}_8$, or $\tilde{E}_8$. In all cases there are only nine (-2)-curves on the surface;

(2) $\overline{-K_X}$ can only be written in one manner as strictly positive linear combination of the nine (-2)-curves.

In fact these properties are consequences of the following two facts:

(1) if zero is a nontrivial linear combination of the set of (-2)-curves, then the surface must be elliptic (see [1, Proposition 1.2, page 26]);

(2) if a divisor is orthogonal to K_X and of self-intersection zero, then it is a multiple of K_X (see [2, Lemma 2]).

Now we consider examples of surfaces with different configurations of (-2)-curves.

Case 1 (the configuration is $\tilde{E}_8$). We have

$$-K_X = C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 + 6C_6 + 4C_7 + 2C_8 + 3C_9. \tag{2.1}$$

Our bound is equal to 1, consequently there is only one (-1)-curve: the exceptional divisor of the last blowup.

Case 2 (the configuration is $\tilde{D}_8$). We have

$$-K_X = C_1 + C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2C_7 + C_8 + C_9. \tag{2.2}$$

Using Lemma 1.2 and Corollary 2.2, we deduce that the number of (-1)-curves is at most 4, whereas our bound is 15.

Case 3 (the configuration is $\tilde{A}_8$). We have

$$-K_X = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9. \tag{2.3}$$

Using Lemma 1.2 and Corollary 2.2, we deduce that the number of (-1)-curves is at most 9, whereas our bound is 255.

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