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ON THE FIRST POWER MEAN OF L-FUNCTIONS WITH THE WEIGHT OF GENERAL KLOOSTERMAN SUMS

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The main purpose of this paper is using the estimates for character sums and the analytic method to study the first power mean of Dirichlet L-functions with the weight of general Kloosterman sums, and give an interesting asymptotic formula.

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1. Introduction. Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q. For any integers m and n, we define the general Kloosterman sums $S(m,n,\chi,q)$ as follows:

$$S(m,n,\chi,q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{ma+n\bar{a}}{q}\right), \tag{1.1}$$

where $\bar{a}$ denotes the inverse of a modulo q and $e(y) = e^{2\pi i y}$. This summation is very important, because it is a generalization of the classical Kloosterman sums. Many authors had studied the properties of $S(m,n,\chi,q)$. For instance, Chowla [1] and Malyšev [3] obtained a sharper upper bound estimation for $S(m,n,\chi,q)$. That is,

$$|S(m,n,\chi,p)| \ll (m,n,p)^{1/2} p^{1/2+\epsilon}, \tag{1.2}$$

where p is a prime, ϵ is any fixed positive number, and (m,n,p) denotes the greatest common divisor of m, n, and p. But for an arbitrary composite number q, we do not know how large $|S(m,n,\chi,q)|$ is. In fact the value of $|S(m,n,\chi,q)|$ is quite irregular if q is not a prime. The main purpose of this paper is to obtain some good distribution properties of $|S(m,n,\chi,q)|$ in some weight mean value problems. For convenience, in this paper we always suppose $q \geq 3$ be an integer and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to character $\chi \bmod q$. Then we can use the estimates for character sums and the analytic method to prove the following main result.

Theorem 1.1. *For any integers m and n with $(mn,q) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |S(m,n,\chi,q)|^2 \cdot |L(1,\chi)| = C \cdot \phi^2(q) + O\left(q^{3/2} \cdot d^2(q) \cdot \ln^2 q\right), \tag{1.3}$$

where

$$C = {\prod_p}' \left[1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \cdots + \frac{\binom{2n}{n}^2}{4^{2n} \cdot p^{2n}} + \cdots\right] \tag{1.4}$$

is a constant, $\sum_{\chi\neq\chi_0}$ *denotes the summation over all nonprincipal characters modulo* q*,* $\prod_p'$ *denotes the product over all primes* p *with* $(p,q)=1$*,* $d(q)$ *is the divisor function,* $\phi(q)$ *is the Euler function, and* $\binom{2n}{n}=(2n)!/(n!)^2$*.*

For general integer $k\geq 2$, whether there exists an asymptotic formula for

$$\sum_{\chi\neq\chi_0}|S(m,n,\chi,q)|^{2k}|L(1,\chi)| \tag{1.5}$$

is an unsolved problem.

2. Some lemmas. In order to complete the proof of Theorem 1.1, we need the following lemmas.

Lemma 2.1. *For any integer* $q\geq 3$*, we have the estimate*

$$\sum_{d\mid q} d^{1/2}\sum_{s=1}^{q/d-1}\left|\sum_{\chi\neq\chi_0}\chi(sd+1)|L(1,\chi)|\right| = O\left(q\cdot d(q)\cdot\ln^2 q\right). \tag{2.1}$$

Proof. Let $N=q^{3/2}$, χ be a nonprincipal character $\bmod q$ and $A(\chi,y)=\sum_{N<n\leq y}\chi(n)$. Then by Abel identity and Pólya-Vinogradov inequality, we have

$$\begin{aligned} L(1,\chi) &= \sum_{n\leq N}\frac{\chi(n)}{n}+\int_N^{+\infty}\frac{A(\chi,y)}{y^2}dy \\ &= \sum_{n\leq N}\frac{\chi(n)}{n}+O\left(\frac{\ln q}{q}\right). \end{aligned} \tag{2.2}$$

So that

$$|L(1,\chi)| = \left|\sum_{n\leq N}\frac{\chi(n)}{n}\right|+O\left(\frac{\ln q}{q}\right). \tag{2.3}$$

On the other hand, let $r(n)$ be a multiplicative function defining by

$$r(p^\alpha)=\frac{\binom{2\alpha}{\alpha}}{4^\alpha},\qquad r(1)=1, \tag{2.4}$$

where p is a prime and α is any positive integer. For the function $r(n)$, it is easy to prove that

$$\sum_{d\mid n} r(d)\cdot r\left(\frac{n}{d}\right)=1, \tag{2.5}$$

$$\begin{aligned} \left(\sum_{n\leq N}\frac{\chi(n)r(n)}{n}\right)^2 &= \sum_{m\leq N}\sum_{n\leq N}\frac{\chi(nm)r(m)r(n)}{mn} \\ &= \sum_{n\leq N}\frac{\chi(n)}{n}+\sum_{N<n\leq N^2}\frac{\chi(n)r(n,N)}{n}, \end{aligned} \tag{2.6}$$

where

$$r(n,N) = \sum_{\substack{d\mid n \\ d,\ n/d\le N}} r(d)\cdot r\left(\frac{n}{d}\right). \tag{2.7}$$

Note that the triangle inequalities

$$|a|-|b| \le |a\pm b|, \qquad |a\pm b| \le |a|+|b|, \tag{2.8}$$

from (2.3), (2.6), Cauchy inequality, and the orthogonality relationship for character sums

$$\begin{aligned}
\sum_{\chi \bmod q} \chi(n) &= \begin{cases} \phi(q), & \text{if } n\equiv 1 \bmod q; \\ 0, & \text{otherwise,} \end{cases} \\
\sum_{a=1}^{q} \chi(a) &= \begin{cases} \phi(q), & \text{if } \chi=\chi_0; \\ 0, & \text{otherwise,} \end{cases}
\end{aligned} \tag{2.9}$$

we have

$$\begin{aligned}
&\sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi\ne\chi_0} \chi(sd+1)\,|L(1,\chi)|\right| \\
&= \sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi\ne\chi_0} \chi(sd+1) \left| \sum_{n\le N} \frac{\chi(n)}{n}\right| + O(\ln q) \right| \\
&= \sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left| \sum_{n\le N} \frac{\chi(n)}{n}\right| \right| + O(q\cdot d(q)\cdot \ln q) \\
&\ll \sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left(\left| \sum_{n\le N} \frac{\chi(n)}{n}\right| - \left| \sum_{n\le N} \frac{\chi(n)r(n)}{n}\right|^2 \right) \right| \\
&\quad + \sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left| \sum_{n\le N} \frac{\chi(n)r(n)}{n}\right|^2 \right| + q\cdot d(q)\cdot \ln q \\
&\ll \sum_{d\mid q} d^{1/2}\cdot \frac{q}{\sqrt{d}} \cdot \left[\sum_{\chi} \left(\left| \sum_{n\le N} \frac{\chi(n)}{n}\right| - \left| \sum_{n\le N^2} \frac{\chi(n)r(n,N)}{n}\right| \right)^2 \right]^{1/2} \\
&\quad + \phi(q) \sum_{d\mid q} d^{1/2} \sum_{s=1}^{q/d-1} \sideset{}{'}\sum_{\substack{m\le N \\ (sd+1)m\equiv n \bmod q}} \sideset{}{'}\sum_{n\le N} \frac{r(m)r(n)}{mn} + q\cdot d(q)\cdot \ln q \\
&\ll q\cdot d(q)\cdot \left[\sum_{\chi} \left| \sum_{N<n\le N^2} \frac{\chi(n)r(n,N)}{n}\right|^2 \right]^{1/2} + q\cdot d(q)\cdot \ln q \\
&\quad + \phi(q) \sum_{d\mid q} d^{1/2} \sideset{}{'}\sum_{\substack{m\le N \\ m\equiv n \bmod d \\ m\ne n}} \sideset{}{'}\sum_{n\le N} \frac{r(m)r(n)}{mn}
\end{aligned}$$

$$\ll q^{3/2}\cdot d(q)\cdot\left(\sum_{N<m\le N^2}{}' \sum_{\substack{N<n\le N^2\\ m\equiv n \bmod q}}{}' \frac{r(m,N)\cdot r(n,N)}{mn}\right)^{1/2} + q\cdot d(q)\cdot \ln^2 q$$
$$\ll q\cdot d(q)\cdot\ln^2 q. \tag{2.10}$$

This proves Lemma 2.1. □

Lemma 2.2. *For any integer $q\ge 3$, we have the asymptotic formula*

$$\sum_{\chi\ne\chi_0}\left|L(1,\chi)\right| = C\cdot\phi(q)+O\left(q^{1/2}\cdot\ln q\right), \tag{2.11}$$

where

$$C=\sum_{n=1}^{\infty}{}'\frac{r^2(n)}{n^2}=\prod_p{}'\left[1+\frac{\binom{2}{1}^2}{4^2\cdot p^2}+\frac{\binom{4}{2}^2}{4^4\cdot p^4}+\cdots+\frac{\binom{2n}{n}^2}{4^{2n}\cdot p^{2n}}+\cdots\right] \tag{2.12}$$

is an absolute constant.

Proof. Let $N=q^{3/2}$. Then from (2.3), (2.6), the orthogonality relationship for character sums, and the method of proving Lemma 2.1, we have

$$\begin{aligned}
\sum_{\chi\ne\chi_0}\left|L(1,\chi)\right| &= \sum_{\chi\ne\chi_0}\left|\sum_{n\le N}\frac{\chi(n)}{n}\right| + O(\ln q)\\
&= \sum_{\chi}\left|\sum_{n\le N}\frac{\chi(n)r(n)}{n}\right|^2 + O(\ln q)\\
&\quad + \sum_{\chi}\left(\left|\sum_{n\le N}\frac{\chi(n)}{n}\right| - \left|\sum_{n\le N}\frac{\chi(n)r(n)}{n}\right|^2\right)\\
&= \phi(q)\cdot\sum_{m\le N}{}'\sum_{\substack{n\le N\\ m\equiv n \bmod q}}{}'\frac{r(m)\cdot r(n)}{mn} + O(\ln q)\\
&\quad + O\left(\sum_{\chi}\left|\sum_{N<n\le N^2}\frac{\chi(n)\cdot r(n,N)}{n}\right|\right)\\
&= \phi(q)\cdot\sum_{\substack{n=1\\ (n,q)=1}}^{\infty}\frac{r^2(n)}{n^2} + O\left(\ln^2 q\right)\\
&\quad + O\left(q^{1/2}\left(\sum_{\chi}\left|\sum_{N<n\le N^2}\frac{\chi(n)\cdot r(n,N)}{n}\right|^2\right)^{1/2}\right)\\
&= \phi(q)\cdot\sum_{\substack{n=1\\ (n,q)=1}}^{\infty}\frac{r^2(n)}{n^2} + O\left(q^{1/2}\cdot\ln q\right)\\
&= C\cdot\phi(q)+O\left(q^{1/2}\cdot\ln q\right).
\end{aligned} \tag{2.13}$$

This proves Lemma 2.2. □

Lemma 2.3. *Let m, n, and q be integers with $q \geq 3$. Then we have the estimates*

$$S(m,n,q) = \sum_{a=1}^{q}{}' e\left(\frac{ma+n\bar{a}}{q}\right) \ll (m,n,q)^{1/2} q^{1/2} d(q), \tag{2.14}$$

where $\sum_a'$ denotes the summation over all a such that $(a,q) = 1$.

Proof. See [2]. □

3. Proof of Theorem 1.1.

In this section, we will complete the proof of Theorem 1.1. First we have

$$\begin{aligned}
&\sum_{\chi \neq \chi_0} |L(1,\chi)| \cdot |S(m,n,\chi,q)|^2 \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' e\left(\frac{(r-s)m+(\bar{r}-\bar{s})n}{q}\right) \sum_{\chi \neq \chi_0} \chi(r\bar{s}) |L(1,\chi)| \\
&= \sum_{r=1}^{q}{}' \sum_{s=1}^{q}{}' e\left(\frac{r(1-\bar{s})m+\bar{r}(1-s)n}{q}\right) \sum_{\chi \neq \chi_0} \chi(s) |L(1,\chi)| \\
&= \phi(q) \sum_{\chi \neq \chi_0} |L(1,\chi)| + \sum_{s=2}^{q}{}' \sum_{r=1}^{q}{}' e\left(\frac{r(1-\bar{s})m+\bar{r}(1-s)n}{q}\right) \sum_{\chi \neq \chi_0} \chi(s) |L(1,\chi)|.
\end{aligned} \tag{3.1}$$

From (3.1) and Lemmas 2.2 and 2.3 we get

$$\begin{aligned}
&\sum_{\chi \neq \chi_0} |L(1,\chi)| \cdot |S(m,n,\chi,q)|^2 \\
&\quad = C \cdot \phi^2(q) + O\left(q^{3/2} \cdot \ln q\right) \\
&\quad\quad + O\left(q^{1/2} d(q) \sum_{s=2}^{q}{}' ((s-1)m, (\bar{s}-1)n, q)^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(s) |L(1,\chi)| \right| \right).
\end{aligned} \tag{3.2}$$

Note that $(m,q) = (n,q) = (s,q) = 1$, and $(1-s,q) = (s\bar{s}-s,q) = (s(\bar{s}-1),q) = (\bar{s}-1,q)$, so we have $(s-1,\bar{s}-1,q) = (s-1,q)$. Thus from (3.2) and Lemma 2.1 we obtain the asymptotic formula

$$\begin{aligned}
&\sum_{\chi \neq \chi_0} |L(1,\chi)| \cdot |S(m,n,\chi,q)|^2 \\
&\quad = C \cdot \phi^2(q) + O\left(q^{3/2} \cdot \ln q\right) \\
&\quad\quad + O\left(q^{1/2} d(q) \sum_{s=2}^{q}{}' (s-1,q)^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(s) |L(1,\chi)| \right| \right) \\
&\quad = C \cdot \phi^2(q) + O\left(q^{3/2} \cdot \ln q\right) \\
&\quad\quad + O\left(q^{1/2} d(q) \sum_{d \mid q} \sum_{t=1}^{q/d-1} d^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(td+1) |L(1,\chi)| \right| \right) \\
&\quad = C \cdot \phi^2(q) + O\left(q^{3/2} \cdot d^2(q) \cdot \ln^2 q\right).
\end{aligned} \tag{3.3}$$

This completes the proof of the theorem.

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References

[1] S. Chowla, *On Kloosterman's sum*, Norske Vid. Selsk. Forh. (Trondheim) **40** (1967), 70–72.
[2] T. Estermann, *On Kloosterman's sum*, Mathematika **8** (1961), 83–86.
[3] A. V. Malyšev, *A generalization of Kloosterman sums and their estimates*, Vestnik Leningrad. Univ. **15** (1960), no. 13, 59–75 (Russian).

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