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POWERS OF COMMUTATORS AS PRODUCTS OF SQUARES

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Let F be a free group and x, y be two distinct elements of a free generating set, then $[x,y]^n$ is not a product of two squares in F, and it is the product of three squares. We give a short combinatorial proof.

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1. Introduction. It has been shown by Lyndon and Newman [2] that in the free group $F = F(x,y)$, freely generated by x, y, the commutator $[x,y]$ is never the product of two squares in F, although it is always the product of three squares. Let $\gamma \in F'$, the *minimal number of squares which is required to write* γ *as a product of squares in* F is called the square length of γ and denoted by $\mathrm{Sq}(\gamma)$. Here we consider more general case, that is, $\mathrm{Sq}[x,y]^n$, $n \in \mathbb{N}$.

Throughout this paper, x^y means yxy^{-1}; $[x,y] = xyx^{-1}y^{-1}$; G' denotes the derived subgroup of G, and $\gamma_m(G)$ denotes the mth term of the lower central series of G.

2. Main result. The main result of this note is the following theorem.

Theorem 2.1. *Let F be a free group and let x, y be two distinct elements of a free generating set, then* $\mathrm{Sq}[x,y]^n = 3$ *if $n \in \mathbb{N}$ is odd, and* $\mathrm{Sq}[x,y]^n = 1$ *if n is even.*

Proof. In the case when n is even, the result is clear. Let n be an odd integer. First, we show that $[x,y]^n$ can be written as a product of 3 squares in F. Put $[x,y] = W$, then we can check the following identity:

$$W^{2k+1} = [x,y]^{2k+1} = \left(\left(W^k xy\right)^{W^k}\right)^2 \left(W^k y^{-1}\right)^2 \left(\left(W^{-k}x^{-1}\right)^y\right)^2. \tag{2.1}$$

In the case $k = 0$, we get

$$[x,y] = (xy)^2 \left(y^{-1}\right)^2 \left(\left(x^{-1}\right)^y\right)^2, \tag{2.2}$$

hence

$$\mathrm{Sq}[x,y]^n \le 3, \tag{2.3}$$

hence to complete the proof it is enough to show that

$$\mathrm{Sq}[x,y]^n \ne 2. \tag{2.4}$$

The case $n = 1$ was proved by Lyndon and Newman [2], so we prove that $W^{2k+1} \neq a^2b^2$ for any $k \in \mathbb{N}$ and $a, b \in F$. Lyndon and Schützenberger [3] proved that

$$a^M = b^N c^P, \quad M, N, P \geq 2, \tag{2.5}$$

implies that a, b, and w all lie in a cyclic subgroup. Therefore, all components a, b, and w of a solution of the equation $W^r = a^2b^2$, for $r \geq 2$, must belong to the cyclic subgroup generated by W. Hence, we reduce the problem to the case of rank two, we may assume $F = F(x, y)$ to be the free group of rank two freely generated by x, y, and suppose $a^2b^2 = W^r$ for some $r \in \mathbb{Z}$, then

$$a^2b^2 \equiv (ab)^2 \operatorname{mod} F'. \tag{2.6}$$

Since $a^2b^2 \in F'$, $(ab)^2 \in F'$, hence $ab \in F'$ and $a = ub^{-1}$ for some $u \in F'$. Now $a^2 = (ub^{-1})^2 = uu^{b^{-1}}b^{-2}$, hence $uu^{b^{-1}} = W^r$ and $W^r \equiv u^2 (\operatorname{mod} \gamma_3(F))$.

But $\gamma_2(F)/\gamma_3(F) \cong C_\infty$ and it is generated by $W = [x, y]$. Since W is the generator of $\gamma_2(F) \operatorname{mod} \gamma_3(F)$, $u^2 \equiv W^r$ has solution if and only if r is even, hence we proved that $W^{2k+1} \neq a^2b^2$ for any $k \in \mathbb{N}$. □

We have the following notations.

(1) In a similar way $a^nb^n = W^r$ for some $r \in \mathbb{Z}$ implies that

$$\begin{gathered} a^n = (ub^{-1})^n = uu^{b^{-1}}u^{b^{-2}} \cdots u^{b^{-(n-1)}} b^{-n}, \\ a^nb^n = uu^{b^{-1}}u^{b^{-2}} \cdots u^{b^{-(n-1)}}, \end{gathered} \tag{2.7}$$

for some $u \in F'$. And we have

$$u^n \equiv W^r \operatorname{mod} \gamma_3(F), \tag{2.8}$$

so, $n \mid r$, hence, if n is not a multiple of r, then $a^nb^n \neq W^r$.

(2) In $F(x, y)$, $\operatorname{Sq}[x, y]^n = 3$ for any odd number $n \in \mathbb{N}$. But there exists commutators with square length equals to two. Obviously, $[h^2, g]$ and $[h, g^2]$ are products of two squares, and a nontrivial commutator is never a square [4]. Thus $\operatorname{Sq}[h^2, g] = \operatorname{Sq}[h, g^2] = 2$.

But it is not the only case in which the square length of a commutator is two, as shown by Comerford and Edmundss in [1].

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