THE HAUSDORFF DIMENSION AND EXACT HAUSDORFF MEASURE OF RANDOM RECURSIVE SETS WITH OVERLAPPING

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We weaken the open set condition and define a finite intersection property in the construction of the random recursive sets. We prove that this larger class of random sets are fractals in the sense of Taylor, and give conditions when these sets have positive and finite Hausdorff measures, which in certain extent generalize some of the known results, about random recursive fractals.

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1. Introduction. As it is known, the separation conditions, such as the strong separation condition, the open set condition (OSC), and the strong open set condition, must be taken into consideration when computing the Hausdorff dimensions of the random recursive sets. In deterministic cases, Schief [13] proved that the strong open set condition and the open set condition are both equivalent to $\infty > \mathcal{H}^\alpha(K) > 0$, where K is the strictly self-similar set (cf. Hutchinson [9]) in $\mathbb{R}^d$, α is the similarity dimension of K, and $\mathcal{H}^\alpha$ denotes the Hausdorff measure of this dimension. But in random cases, we do not have such good results, many authors, such as Cawley and Mauldin [2], Falconer [3], Graf [6], Mauldin and Williams [12], Arbeiter and Patzschke [1], and Hu [7, 8], have discussed the fractal properties of the random recursive set $K(\omega)$, and the most general result may be: if the open set condition is satisfied in the random recursive process of i.i.d. contraction similitudes, then $\dim K(\omega) = \alpha$ with probability one, where α is the unique solution of the equation

$$\mathbf{E} \sum_{i=1}^N r_i^\alpha = 1, \tag{1.1}$$

and $\mathbf{E}$ is the expectation operator and r_i is the Lipschitz coefficients of the similitudes.

Sometimes the open set condition in the construction of recursive sets is complex and difficult to verify. In this paper, we try to find another criterion to calculate the fractal dimensions of some random recursive sets, we give a definition of the finite intersection property (FIP) which allows appropriate overlapping in the same level. This condition is rather easy to verify, especially in the generalized Moran sets and Mauldin-Williams (M-W) models [12] (in fact, the open set condition is equivalent to the nonoverlapping in the same level in the recursive process of M-W models). We prove that if the recursive process satisfies the OSC, then it satisfies the FIP, and we give examples which satisfy FIP but do not satisfy OSC; we also prove the following theorem.

THEOREM 1.1. *For every $\epsilon > 0$, if $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \ \sigma \in D\} \subset \mathrm{sicon}(\Omega, \mathbb{R}^d)^N$ satisfies ϵ-FIP, and $\{(r_{\sigma*1},\ldots,r_{\sigma*N}), \ \sigma \in D\}$ is a family of i.i.d. random elements, then*

$$\mathrm{Dim}\,K(\omega) = \dim K(\omega) = \min\{\alpha, d\} \quad a.e. \tag{1.2}$$

still holds; furthermore if $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \ \sigma \in D\} \subset \mathrm{sicon}(\Omega, \mathbb{R}^d)^N$ satisfies 0-FIP, if $\sum_{i=1}^N r_i^\alpha = 1$ a.e. and $\alpha \le d$, then $\infty > \mathscr{H}^\alpha(K) > 0$ a.e., where α is defined by (1.1), $\dim$ and Dim stand for Hausdorff dimension and Packing dimension, respectively, and $\mathscr{H}^\alpha$ is the Hausdorff measure.

2. Preliminaries. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, and let (E, ρ) be a separable complete metric space. For all $f : E \mapsto E$, we call

$$\mathrm{Lip}(f) = \sup_{x \ne y, x, y \in \mathbf{E}} \frac{\rho(f(x), f(y))}{\rho(x, y)} \tag{2.1}$$

the Lipschitz coefficient of f. Denote

$$\mathrm{con}(E) = \{f : \mathrm{Lip}(f) < 1, \ f : E \longmapsto E\}. \tag{2.2}$$

In this paper, we take E to be $\mathbb{R}^d$, the Euclidean space. Let $\mathrm{con}(\mathbb{R}^d)$ be equipped with the usual topology of uniform convergence on compact sets. Let $\mathrm{sicon}(\mathbb{R}^d)$ be the space of all contraction similitudes, and let $\mathrm{sicon}(\Omega, \mathbb{R}^d)$ be the space of all random contraction similitudes.

Let N be a positive integer and let $C_n = \{1,\ldots,N\}^n$. Denote by $D = \bigcup_{n=0}^\infty C_n$ the family of all finite sequences $\sigma = (\sigma_1,\ldots,\sigma_n)$ in $\{1,\ldots,N\}$, by $|\sigma| = n$ the length of $\sigma \in C_n$, by $\tau|n = (\tau_1,\ldots,\tau_{n \wedge |\tau|})$ the curtailment for $\tau \in D$, by $\sigma * \tau = (\sigma_1,\ldots,\sigma_{|\sigma|}, \tau_1,\ldots,\tau_{|\tau|})$ the juxtaposition of σ and τ, by C the infinite sequences $\sigma = (\sigma_1, \sigma_2,\ldots)$, and by $[\sigma] = \{\tau \in C, \tau \mid |\sigma| = \sigma\}$ the cylinder in C. For brevity, write $S_{(|\sigma|, \sigma)} = S_{\sigma|1} \circ \cdots \circ S_\sigma$, $S_{(0, \varnothing)} = I$, where $S_\tau \in \mathrm{sicon}(\Omega, \mathbb{R}^d)$ for $\tau \in D$, and I is the identity; $r_\sigma = \mathrm{Lip}\,S_\sigma$ and $\mathbf{r}_{(|\sigma|, \sigma)} = r_{\sigma|1} \cdots r_\sigma$.

Throughout the paper, we suppose that $r \hat{=} \min_{\sigma \in C_n, n \ge 1} \mathrm{essinf}\,\mathrm{Lip}(S_\sigma) > 0$ and $\max_{\sigma \in C_n, n \ge 1} \mathrm{ess\,sup}\,\mathrm{Lip}(S_\sigma) < 1$, and E is a fixed nonempty compact subset of $\mathbb{R}^d$. For a set $J \subset E$, let J^0 denote the set of interior points of J.

DEFINITION 2.1. Let $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \ \sigma \in D\}$ be a collection of random elements from $(\Omega, \mathscr{F}, P)$ to $\mathrm{con}(\mathbb{R}^d)^N$. If there exists a nonempty open set O in E, such that for P-a.e. ω,

    (i) $S_{(|\sigma|+1, \sigma*i)}(O) \subset S_{(|\sigma|, \sigma)}(O)$, for all $i = 1,\ldots,N$, $\sigma \in D$.

    (ii) $S_{(|\sigma|+1, \sigma*i)}(O) \cap S_{(|\sigma|+1, \sigma*j)}(O) = \varnothing$ for all $i \ne j$, $\sigma \in D$.

Then the family of $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \ \sigma \in D\}$ satisfies the open set condition (OSC).

For any $\epsilon \ge 0$, we say that the family $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \ \sigma \in D\}$ satisfies the finite intersection property of level ϵ (in short, ϵ-FIP) if there exists a nonempty compact set J in E with $\mathrm{diam}(J) = \theta > 0$, $\overline{J^0} = J$, and a number $e > 1$ such that for P-a.e. ω,

    (iii) $J_{(|\sigma|+1, \sigma*i)} \subset J_{(|\sigma|, \sigma)}$ for all $i = 1, 2,\ldots,N$ and $\sigma \in D$.

(iv) There exists $\delta \hat{=} \delta(\epsilon) > 0$, for every $M \subset J$ with $\operatorname{diam}(M) < \min\{r, \delta\}$, we have

$$\operatorname{Card} H_M^\omega \hat{=} \operatorname{Card}\left\{\sigma^\omega \in D : r^{k+1}\theta < \operatorname{diam}\left(J_{(|\sigma|,\sigma)}\right) \le r^k\theta,\ J_{(|\sigma|,\sigma)} \cap M \ne \varnothing\right\}$$
$$< e(\operatorname{diam} M)^{-\epsilon}, \tag{2.3}$$

where $J_{(|\sigma|,\sigma)} = S_{(|\sigma|,\sigma)}(J)$, $J_{(0,\varnothing)} = J$, $k = \max\{n : \operatorname{diam}(M) \le r^n\theta\}$.

REMARK 2.2. It is clear that if $0 \le \epsilon_1 < \epsilon_2$ then ϵ_1-FIP implies ϵ_2-FIP.

REMARK 2.3. If in the definition we have

$$\limsup_{\operatorname{diam}(M)\to 0} \frac{\operatorname{Card} H_M}{k} < \infty, \tag{2.4}$$

then, for all $\epsilon > 0$, the family $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\}$ satisfies the ϵ-FIP. Furthermore, if

$$0 < \limsup_{\operatorname{diam}(M)\to 0} \frac{\operatorname{Card} H_M}{k} < \infty \tag{2.5}$$

exists, then the family $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\}$ does not satisfy 0-FIP.

For simplicity, we write FIP for 0-FIP. Note that if $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\}$ is i.i.d., then $K(\omega)$ is a statistically self-similar set (cf. [7, 8]). However, if $\{(r_{\sigma*1}, \ldots, r_{\sigma*N}),\ \sigma \in D\}$ is i.i.d., then $K(\omega)$ need not be a statistically self-similar set [2].

LEMMA 2.4. *If the family of* $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\} \subset \operatorname{sicon}(\Omega, E)^N$ *satisfies the OSC, then it satisfies the FIP.*

PROOF. Suppose that $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\}$ satisfies the OSC, and O is defined as in (i) and (ii) in Definition 2.1. Take $J = \bar{O}$, since $S_\tau(O) \subset O$ for every $\tau \in D$, then

$$J_{(|\sigma|+1,\sigma*i)} = S_{(|\sigma|,\sigma)}(S_{\sigma*i}(J)) = S_{(|\sigma|,\sigma)}\left(\overline{S_{\sigma*i}(O)}\right)$$
$$\subset S_{(|\sigma|,\sigma)}(\bar{O}) = J_{(|\sigma|,\sigma)}, \tag{2.6}$$

this means that (iii) is true.

Without loss of generality, we always assume that $\operatorname{diam}(J) = 1$. Let $M \subset J$, $\operatorname{diam}(M) < r$, by the definition of k, we have $\operatorname{diam}(M) \le r^k$. For any $\sigma \in H_M$, we have

$$\operatorname{diam}\left(J_{(|\sigma|,\sigma)}\right) \le r^k, \qquad J_{(|\sigma|,\sigma)} \cap M \ne \varnothing. \tag{2.7}$$

Hence, there is a closed ball $B(x, 2r^k)$ with center x in M and radius $2r^k$ such that

$$\bigcup_{\sigma \in H_M} J_{(|\sigma|,\sigma)} \subset B(x, 2r^k). \tag{2.8}$$

Now we want to show that, for any $\sigma, \tau \in H_M$ with $\sigma \ne \tau$, $J^0_{(|\tau|,\tau)} \cap J^0_{(|\sigma|,\sigma)} = \varnothing$. Note that, by the definition of H_M, if $\sigma, \tau \in H_M$, $\sigma \ne \tau$, then σ and τ are incomparable, that is, neither $\sigma \mid |\tau| = \tau$ nor $\tau \mid |\sigma| = \sigma$ holds.

Write $\sigma \wedge \tau$ being the longest sequence $\gamma \in D$ such that $\gamma = \sigma \mid t = \tau \mid t$ for some integer k. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\tau = (\tau_1, \ldots, \tau_m)$, $l = |\sigma \wedge \tau| < \min\{n, m\}$, then

$$J^0_{(|\sigma|,\sigma)} \cap J^0_{(|\tau|,\tau)} = S_{(l,\sigma|l)}\Big(\big[S_{\sigma|(l+1)} \circ \cdots \circ S_{\sigma|n}(O)\big] \cap \big[S_{\tau|(l+1)} \circ \cdots \circ S_{\tau|m}(O)\big]\Big)$$
$$\subset S_{(l,\sigma|l)}\Big(S_{\sigma|(l+1)}(O) \cap S_{\tau|(l+1)}(O)\Big) = \varnothing. \tag{2.9}$$

Then, by

$$\mathrm{diam}\,(J_{(|\sigma|,\sigma)}) = \mathrm{diam}\,(S_{(|\sigma|,\sigma)}(J)) = \mathrm{Lip}\,(S_{(|\sigma|,\sigma)}),$$
$$\mathscr{L}(J_{(|\sigma|,\sigma)}) = \mathrm{Lip}\,(S_{(|\sigma|,\sigma)})^d \mathscr{L}(J), \tag{2.10}$$

we have

$$\mathscr{L}\Bigg(\bigcup_{\sigma \in H_M} J_{(|\sigma|,\sigma)}\Bigg) \geq \mathscr{L}\Bigg(\bigcup_{\sigma \in H_M} J^0_{(|\sigma|,\sigma)}\Bigg) = \sum_{\sigma \in H_M} \mathscr{L}(J_{(|\sigma|,\sigma)}) \geq (\mathrm{Card}\,H_M)(r^{k+1})^d \mathscr{L}(J). \tag{2.11}$$

But

$$\mathscr{L}\Bigg(\bigcup_{\sigma \in H_M} J_{(|\sigma|,\sigma)}\Bigg) \leq \mathscr{L}(B(x, 2r^k)) \leq (2r^k)^d V_d, \tag{2.12}$$

where $\mathscr{L}$ is the Lebesgue measure on $\mathbb{R}^d$ and V_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. So,

$$\mathrm{Card}\,H_M \leq \frac{(2r^k)^d V_d}{(r^{k+1})^d \mathscr{L}(J)} \tag{2.13}$$

and [Lemma 2.4](#) is proved. $\qquad\square$

3. Main results. We always assume that $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}),\ \sigma \in D\} \subset \mathrm{sicon}(\Omega, E)^{\mathcal{N}}$ and let $\mathcal{N} = \{1, 2, \ldots\}$ be the set of positive integers in this section. For almost every $\omega \in \Omega$, for any fixed positive integer $i \in \mathcal{N}$, we define a stopping $\gamma^\omega_i : C \mapsto \mathcal{N}$ by assigning to each $\sigma \in C$ the value

$$\gamma^\omega_i(\sigma) = \min\{n : r_{\sigma|1} \cdot r_{\sigma|2} \cdots r_{\sigma|n} \leq r^i\} = \{n : \mathrm{Lip}\,(S_{(n,\sigma|n)}) \leq r^i\}, \tag{3.1}$$

and let

$$\Gamma_i(\omega) = \{\tilde{\sigma}_i \,\hat{=}\, \sigma \mid \gamma^\omega_i(\sigma) : \sigma \in C\}, \tag{3.2}$$

then for $\sigma \in C$, we have $r^{i+1} < \mathrm{diam}(J_{(|\tilde{\sigma}_i|, \tilde{\sigma}_i)}) = \mathrm{Lip}(S_{(|\tilde{\sigma}_i|, \tilde{\sigma}_i)}) \leq r^i$. Note that for every $i \in \{1, 2, \ldots\}$, we have

$$\mathbf{E}\Bigg(\sum_{\sigma \in \Gamma_i} r^\alpha_{(\sigma, |\sigma|)}\Bigg) = 1. \tag{3.3}$$

PROPOSITION 3.1. *If* $\{(S_{\sigma*1},\ldots,S_{\sigma*N}),\ \sigma \in D\} \subset \mathrm{sicon}(\Omega,\mathbb{R}^d)^N$ *satisfies condition (i) in Definition 2.1, and* $\{(r_{\sigma*1},\ldots,r_{\sigma*N}),\ \sigma \in D\}$ *is a family of i.i.d. random elements, then*

$$\mathrm{Dim}\,K(\omega) \leq \min\{\alpha,d\} \quad a.e., \tag{3.4}$$

where $K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} S_{n,\sigma}(E)$, *and* α *is the unique solution of (1.1).*

PROOF. Let J be the nonempty compact subset of E such that $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \sigma \in D\}$ is of FIP. First, we show that

$$K(\omega) = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in C_n} S_{n,\sigma}(J). \tag{3.5}$$

In fact, for $\sigma \in C$, $\lim_{n\to\infty} S_{n,\sigma|n}(E) = \{x_\sigma\}$ is a singleton, and $\lim_{n\to\infty} S_{n,\sigma|n}(J) = \{y_\sigma\}$ is a singleton also, and since $J \subset E$, we have $x_\sigma = y_\sigma$, so (3.5) holds.

Define a metric $\rho^*(\sigma,\tau) \hat{=} r^{(|\sigma \wedge \tau|)}$, then (C,ρ^*) is a complete, separable, compact, and totally disconnected metric space [5]. A random code map $\pi^\omega : C \mapsto K(\omega)$ is defined by $\pi(\sigma) = x_\sigma$, then for P-a.e. $\omega\,\pi$ is continuous and $\pi(C) = K$.

For all $\delta \in (0,r)$, $k_0 \hat{=} \min\{k : r^{k+1} < 2\delta\}$, let $\{B(x_i,\delta),\ i = 1,2,\ldots,N(\delta)\}$ be a collection of balls with center x in K and radius δ (note that $N(\delta)$ is finite), then there exist $\sigma\langle i\rangle \in C$, such that $x_i = \pi(\sigma\langle i\rangle), i = 1,2,\ldots$. Note that $\{B(x_i,\delta)\}$ are disjoint. For each $\sigma\langle i\rangle$, there is $\overline{\sigma\langle i\rangle}_{k_0+1}$, the curtailment of $\sigma\langle i\rangle$, such that $r^{k_0+2} < \mathbf{r}_{(|\overline{\sigma\langle i\rangle}_{k_0+1}|,\overline{\sigma\langle i\rangle}_{k_0+1})} \leq r^{k_0+1}$. If $x_i \neq x_j$, then $\overline{\sigma\langle i\rangle}_{k_0+1} \neq \overline{\sigma\langle j\rangle}_{k_0+1}$, otherwise, if $\overline{\sigma\langle i\rangle}_{k_0+1} = \overline{\sigma\langle j\rangle}_{k_0+1}$ we have $J_{(|\overline{\sigma\langle i\rangle}_{k_0+1}|,\overline{\sigma\langle i\rangle}_{k_0+1})} = J_{(|\overline{\sigma\langle j\rangle}_{k_0+1}|,\overline{\sigma\langle j\rangle}_{k_0+1})}$, and $2\delta > r^{k_0+1} \geq \mathrm{diam}(J_{(|\overline{\sigma\langle i\rangle}_{k_0+1}|,\overline{\sigma\langle i\rangle}_{k_0+1})}) \geq \mathrm{dist}(\pi(\sigma),\pi(\sigma)) = 2\delta$, this leads to a contradiction.

Hence, we have

$$\mathbf{E}(N(\delta)r^{(k_0+2)(\alpha+\epsilon)}) \leq \mathbf{E}\left(\sum_i \mathrm{diam}\,(B(x_i,\delta))^{\alpha+\epsilon} r^{\alpha+\epsilon}\right)$$

$$\leq \mathbf{E}\left(\sum_{\overline{\sigma\langle i\rangle}_{k_0+1}} \in \Gamma_{k_0+1}\mathbf{r}_{(|\overline{\sigma\langle i\rangle}_{k_0+1}|,\overline{\sigma\langle i\rangle}_{k_0+1})}^{\alpha+\epsilon}\right) \tag{3.6}$$

$$\leq \sum_{k=1}^{\infty} \mathbf{E}\left(\sum_{i=1}^N r_i^{\alpha+\epsilon}\right)^k < \infty$$

thus for P-a.e. ω,

$$\limsup_{\delta\to 0} \frac{\log N(\delta)}{-\log\delta} \leq \limsup_{\delta\to 0} \frac{\log r^{(k_0+2)\alpha}}{\log\delta} \leq \alpha + \epsilon \tag{3.7}$$

we obtain that, for almost all ω, $\mathrm{Dim}\,K \leq \alpha + \epsilon$ for arbitrary $\epsilon > 0$, which means that $\mathrm{Dim}\,K \leq \min\{d,\alpha\}$ a.e. $\quad\square$

LEMMA 3.2. *Let* $T_n = \sum_{\sigma \in C_n} \mathbf{r}_{(n,\sigma)}^\alpha$, *then* $Y = \lim_{n\to\infty} T_n$ *exists a.e. and* $\mathbf{E}(Y) = 1$, $\infty > Y > 0$ *a.e.*

PROOF. Let $\mathscr{F}_n \hat{=} \sigma\{r_\sigma, |\sigma| \leq n\}$, then $\{T_n, \mathscr{F}_n\}$ is a martingale, and $\mathbf{E}T_n = 1$ for $n = 1, 2, \ldots$, by the martingale convergence theorem, $Y < \infty$ a.e. exists, and $\mathbf{E}Y = 1$.

The equation $Y_\sigma \hat{=} \lim_{n\to\infty} \sum_{\tau \in C_n} (\prod_{t=1}^n r_{\sigma*(\tau|t)}^\alpha)$ exists for all $\sigma \in D$ by the similar argument as above (cf. [1, 12]), and $\{Y_\sigma, \sigma \in D\}$ have the same distributions as that of Y by the i.i.d. property of $\{(r_{\sigma*1}, \ldots, r_{\sigma*N}), \sigma \in D\}$ and $\{Y_\sigma, \sigma \in C_n\}$ is an i.i.d. family. Note that

$$Y = \sum_{\sigma \in C_n} \mathbf{r}_{(n,\sigma)}^\alpha Y_\sigma \quad \text{a.e.} \tag{3.8}$$

If $P\{Y = 0\} = a$, then $a = P\{Y = \sum_{i=1}^N r_i^\alpha Y_i = 0\} = P\{Y_i = 0\}^N = a^N$, then $a = 0$ or $a = 1$. Since $\mathbf{E}Y = 1$, we have $a = 0$. The lemma is proved. □

LEMMA 3.3. *For P-a.e. ω, there is a random measure ν^ω on C satisfying*

$$\nu([\sigma]) = \mathbf{r}_{(|\sigma|,\sigma)}^\alpha Y_\sigma, \qquad \nu(C) = Y. \tag{3.9}$$

PROOF. By the definition of the metric ρ^*, the cylinder $[\sigma]$ is both open and closed, since

$$[\sigma] = \{\tau \in C : \rho(\tau, \sigma) \leq r^{|\sigma|}\} = \{\tau \in C : \rho(\tau, \sigma) < r^{|\sigma|-1}\} \tag{3.10}$$

for every $\sigma \in D$. Let $\mathscr{B}$ be the collection of all cylinders $[\sigma], \sigma \in D$, and let $\mathscr{A}$ be the collection of a finite union of disjoint cylinders, where by convention the empty union is taken to be the empty set $\varnothing$. Then $\mathscr{A}$ is an algebra. Define a random set function ν^ω by $\nu(\varnothing) = 0$, $\nu([\sigma]) \hat{=} \mathbf{r}_{(|\sigma|,\sigma)}^\alpha Y_\sigma$, for $\sigma \in D$, then by (3.8) we have that

$$\nu([\sigma]) = \sum_{i=1}^N \nu([\sigma*i]), \qquad \nu(C) = Y \quad \text{a.e.,} \tag{3.11}$$

so for almost every ω, the set function ν is well defined. By the compactness of C, it can be easily seen that if $A_n \in \mathscr{A}$ decreases to $\varnothing$, then $A_n = \varnothing$ for n large enough, so that $\nu(A_n)$ decreases to $\varnothing$; this shows that ν is a measure on $\mathscr{A}$. In a natural way, we can extend ν to a Borel measure on $\sigma(\mathscr{B})$ (cf. [11]). □

PROPOSITION 3.4. *If for every $\epsilon > 0$, $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}), \sigma \in D\} \subset \text{sicon}(\Omega, \mathbb{R}^d)^N$ satisfying the ϵ-FIP, $\{(r_{\sigma*1}, \ldots, r_{\sigma*N}), \sigma \in D\}$ is a family of i.i.d. random elements, then $\dim(K) \geq \min\{\alpha, d\}$ a.e.*

PROOF. Let $\mu^\omega \hat{=} \nu^\omega \circ \pi^{-1}$, then μ is a random measure and $\text{supp}(\mu) = K$, $\mu(K) = \nu(C) = Y > 0$ a.e. By Proposition 3.1, we have $\dim(K) \leq d$. For $\alpha \leq d$, if we can prove that for every $\epsilon > 0$, there exists $r > \delta > 0$ and a random variable $\infty > l_1(\omega) > 0$ a.e. such that if $M \subset J$ and $\text{diam}(M) < \delta$, then $\mu(M) \leq l_1 \text{diam}(M)^{\alpha-2\epsilon}$ a.e., then it follows from the mass distribution principle, (cf. Falconer [4]), $\dim K \geq \alpha$ a.e.

In fact,

$$\mu(M) = \mu\Big(M \cap K\Big) = \nu \circ \pi^{-1}\Big(M \cap K\Big) \leq \nu \circ \pi^{-1}\left(\bigcup_{\sigma \in H_M} J_{(|\sigma|,\sigma)}\right)$$

$$\leq e(\operatorname{diam} M)^{-\epsilon} \max_{\sigma \in H_M, \sigma \in D} \nu \circ \pi^{-1}(J_{(|\sigma|,\sigma)}) \tag{3.12}$$

$$= e(\operatorname{diam} M)^{-\epsilon} \cdot \nu\{\tau \in C : \pi(\tau) \in J_{(|\sigma_o|,\sigma_0)}\}$$

by the ϵ-FIP, where $\nu(J_{(|\sigma_0|,\sigma_o)}) = \max\{\nu(J_{(|\gamma|,\gamma)}), \gamma \in H_M\}$. If $\pi(\tau) \in J_{(|\sigma_0|,\sigma_o)}$, then $J_{(|\bar{\tau}_k|,\bar{\tau}_k)} \cap J_{(|\sigma_0|,\sigma_o)} \neq \varnothing$ and $r^{k+1} < \operatorname{diam}(J_{(|\bar{\tau}_k|,\bar{\tau}_k)}) \leq r^k$, so $\bar{\tau}_k \in H_{J_{(|\sigma_0|,\sigma_0)}}$, and by ϵ-FIP again, there are at most $[e \cdot (\operatorname{diam} M)^{-\epsilon}] + 1$ such $\bar{\tau}_k^1, \ldots, \bar{\tau}_k^{[e \cdot (\operatorname{diam} M)^{-\epsilon}]+1}$ satisfying the above property, so

$$\{\tau \in C : \pi(\tau) \in J_{(|\sigma_0|,\sigma_o)}\} \subset \left[\bar{\tau}_k^1\right] \cup \cdots \cup \left[\bar{\tau}_k^{[e \cdot (\operatorname{diam} M)^{-\epsilon}]+1}\right], \tag{3.13}$$

hence

$$\mu(M) \leq 2e^2 (\operatorname{diam} M)^{-2\epsilon} \max_{1 \leq i \leq [e(\operatorname{diam} M)^{-\epsilon}]+1} \nu\left[\bar{\tau}_k^i\right]$$

$$= 2e^2 (\operatorname{diam} M)^{-2\epsilon} \max_{i=1,\ldots,[e(\operatorname{diam} M)^{-\epsilon}]+1} r^\alpha_{(|\bar{\tau}_k^i|,\bar{\tau}_k^i)} Y_{\bar{\tau}_k^i}. \tag{3.14}$$

Since $\bar{\tau}_k^i \in H_{J_{(|\sigma_0|,\sigma_0)}}$, $\sigma_0 \in H_M$, we have $r^{k+1} < \mathbf{r}_{(|\bar{\tau}_k^i|,\bar{\tau}_k^i)} \leq r^k$, therefore,

$$\mu(M) \leq 2e^2 \max_{1 \leq i \leq [e(\operatorname{diam} M)^{-\epsilon}]+1} \frac{Y_{\bar{\tau}_k^i} \operatorname{diam}(M)^{\alpha-2\epsilon}}{r^\alpha} \tag{3.15}$$

take $l_1 = 2e^2 \max_{1 \leq i \leq [e(\operatorname{diam} M)^{-\epsilon}]+1} Y_{\bar{\tau}_k^i}/r^\alpha$, note that $\{Y_\sigma, \sigma \in D\}$ have the same distribution as Y, so $0 < l_1 < \infty$ a.e., hence we have $\mu(M) < l_1 \operatorname{diam}(M)^{\alpha-2\epsilon} < \infty$ a.e. The proposition is proved. □

PROPOSITION 3.5. *If $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}), \sigma \in D\} \subset \operatorname{sicon}(\Omega, \mathbb{R}^d)^N$ satisfies FIP, $\{(r_{\sigma*1}, \ldots, r_{\sigma*N}), \sigma \in D\}$ is a family of i.i.d. random elements, and furthermore $\sum_{i=1}^N r_i^\alpha = 1$ a.e. and $\alpha \leq d$, where α is defined in (1.1), then $\infty > \mathcal{H}^\alpha(K) > 0$ a.e.*

PROOF. Since $\{J_{(n,\sigma)}, \sigma \in C_n\}$ is a covering of K for almost all $\omega \in \Omega$, and $\mathbf{r}_{(n,\sigma)} \to 0$, we have

$$\mathbf{E}\big(\mathcal{H}^\alpha(K)\big) \leq \mathbf{E}\left(\sum_{\sigma \in C_n} \mathbf{r}_{(n,\sigma)}^\alpha\right) = 1 < \infty, \tag{3.16}$$

so $\mathcal{H}^\alpha(K) < \infty$ a.e.

Suppose that $\sum_{i=1}^N r_i^\alpha = 1$ a.e., then $Y = 1$ a.e. by its definition. If $\mathcal{H}^\alpha(K) < 1/l_1$ a.e., then there would be a collection $\mathcal{E}$ of sets each with diameter less than r and covering K such that $\sum_{E \in \mathcal{E}} \operatorname{diam}(E)^\alpha < 1/l_1$ a.e. But taking $\epsilon = 0$ in (3.15),

$$1 > \sum_{E \in \mathcal{E}} l_1 \operatorname{diam}(E)^\alpha \geq \sum_{E \in \mathcal{E}} \mu(E) \geq \mu(K) = 1 \quad \text{a.e.} \tag{3.17}$$

This would lead to a contradiction. So this proposition holds. □

Combining Propositions 3.1, 3.4, and 3.5, we obtain Theorem 1.1.

COROLLARY 3.6. *In the deterministic case with specific maps* $\{S_1, \ldots, S_N\}$, *FIP is equal to the open set condition.*

PROOF. By Proposition 3.5, $\infty > \mathscr{H}^\alpha(K) > 0$, where α is the unique solution of $\sum_{i=1}^N \text{Lip}(S_i)^\alpha = 1$. Combining with Schief's result [13], the corollary is true. $\quad\square$

COROLLARY 3.7. *If* $\{(S_{\sigma*1}, \ldots, S_{\sigma*N}), \ \sigma \in D\} \subset \text{sicon}(\Omega, \mathbb{R}^d)^N$ *satisfies the FIP,* $\{(r_{\sigma*1}, \ldots, r_{\sigma*N}), \ \sigma \in D\}$ *is a family of i.i.d. random elements, then* $\dim\mu = \text{Dim}\,\mu = \alpha$ *a.e., where* $\mu = \nu \circ \pi^{-1}$.

PROOF. For every $B(x, h)$ with center $x \in K$ and radius $h < r$, we have

$$\mu(B(x,h)) \le l_1 h^\alpha \tag{3.18}$$

by (3.15), hence

$$\liminf_{h \to 0} \frac{\ln\mu(B(x,h))}{\ln h} \ge \alpha \quad \text{a.e.} \tag{3.19}$$

On the other hand, there exists $\sigma \in C$, such that $\pi(\sigma) = x$, take $\bar{\sigma}$, the curtailment of σ, such that $(1/2)r^{k+1} < |J_{(|\bar{\sigma}|,\bar{\sigma}_k)}| \le (1/2)r^k$, where $k = \max\{n : h \le r^n\}$, so $J_{(|\bar{\sigma}|,\bar{\sigma})} \subset B$, we have

$$\mu(B(x,h)) = \nu \circ \pi^{-1}(B(x,h)) \ge \nu \circ \pi^{-1}(J_{(\bar{\sigma},\bar{\sigma})}) \ge \nu[\tilde{\sigma}] = r^\alpha_{(|\bar{\sigma}|,\bar{\sigma})} Y_{\bar{\sigma}}, \tag{3.20}$$

thus

$$\lim_{h \to 0} \frac{\ln\mu(B(x,h))}{\ln h} \le \alpha \quad \text{a.e.} \tag{3.21}$$

Therefore, $\dim\mu = \text{Dim}\,\mu = \alpha$ a.e. $\quad\square$

4. Examples.
First, we give an example which satisfies the FIP but not the OSC.

EXAMPLE 4.1. Let $J = [0, 1]$, and let $N = 2$, $r_{\sigma*1} = 1/3$, $r_{\sigma*2} = 1/27$, $\sigma \in D$.

(I) The first two steps:

$$s_1^1(x) = \frac{1}{3}x, \qquad s_2^1(x) = \frac{1}{27}x + \frac{8}{27};$$
$$s_{11}^1(x) = \frac{1}{9}x, s_{12}^1(x) = \frac{1}{81}x + \frac{2}{9}, \qquad s_{21}^1(x) = \frac{1}{81}x + \frac{8}{27}, \tag{4.1}$$
$$s_{22}^1(x) = \frac{1}{27 \times 27}x + \left(\frac{1}{3} - \frac{1}{27 \times 27}\right).$$

Thus we have four basic intervals.

(II) The second two steps: as to the interval $[0, 1/9]$, we repeat the same construction technique as in (I). As to the rest of the three basic intervals, we can easily find maps with ratios $\{r_{\sigma*1}, r_{\sigma*2}\} = \{1/3, 1/27\}$ for $\sigma \in C_3$ and C_4, such that the subsets of all these three basic intervals are disjoint. And so on.

PROOF. Obviously, we cannot find an open set such that $\{(S_{\sigma*1}, S_{\sigma*2}), \ \sigma \in D\}$ satisfies OSC, but FIP holds on it. By our theorem, $\dim(K) = \mathrm{Dim}(K) = \alpha$, where α is the solution of $(1/3)^\alpha + (1/27)^\alpha = 1$.

Then we give an example which satisfies the ϵ-FIP for every $\epsilon > 0$, but does not satisfy the FIP. $\square$

EXAMPLE 4.2. Let $J = [0,1]$, and let $S_1(x) = rx$, $S_2(x) = rx + (1-r)$ for $1/2 < r < 1$, then $\{S_1, S_2\}$ does not satisfy the OSC. But if $r = (5^{1/2} - 1)/2$ is a PV number, then $\{(S_{\sigma*1}, S_{\sigma*2}), \ \alpha \in D\}$ satisfies the FIP, where $D = \bigcup_{n=1}^\infty \{1,2\}^n$.

PROOF. By Lau's and Nagi's result in [10], if r is a PV number, then $\{S_1, S_2\}$ satisfies the weak separation property, that is, there exist $z_0 \in J$ and a positive integer l such that, for any $z = S_{(\tau,|\tau|)}(z_0)$, every closed r^k-ball contain at most l distinct $S_{(|\sigma|,\sigma)}(z)$, $\sigma \in \Gamma_k$, ($S_{(|\sigma|,\sigma)}(z)$ can be repeated, that is, we allow that $S_{(\sigma,|\sigma|)}(z) = S_{(\sigma',|\sigma'|)}(z)$ for $\sigma \neq \sigma'$, $\sigma, \sigma' \in \Gamma_k$).

Denote $J_{(|\sigma|,\sigma)} = S_{(|\sigma|,\sigma)}(J)$ for $\sigma \in D$, Definition 2.1(iii) holds obviously. Without loss of generality, let M be an interval in J with $\mathrm{diam}(M) < \delta$ (δ will be defined in the following proof), $r^{k+1} < \mathrm{diam}(M) \leq r^k$, H_M is the set given in Definition 2.1, let $B = B(x, 2r^k)$ be an interval with center $x \in M$ and radius $2r^k$, then $\bigcup_{\sigma \in H_M} J_{(|\sigma|,\sigma)} \subset B$ and $M \subset B$. According to [10, Remark 3], "under the weak separation property, every hr^k-ball with $h > 0$ contains at most $l[2h]^d$ distinct $S_{(|\sigma|,\sigma)}(z)$, $\sigma \in \Gamma_k$," for $z_0 \in J$ and $\sigma \in H_M$, there are at most $4l$ distinct $S_{(|\sigma|,\sigma)}(z_0) \in B$ $\sigma \in H_M \subset \Gamma_k$ (note that $\Gamma_k = C_k$ in this example). If $\sigma\langle 1 \rangle \neq \sigma\langle 2 \rangle \neq \cdots \in H_M$, but $S_{(|\sigma\langle 1 \rangle|,\sigma\langle 1 \rangle)}(z_0) = S_{(|\sigma\langle 2 \rangle|,\sigma\langle 2 \rangle)}(z_0) = \cdots$, then for $z_1 \neq z_0$, $z_1 = S_{(|\tau|,\tau)}(z_0)$ for some $\tau \in D$, we have at most $4l$ distinct $S_{(|\sigma\langle 1 \rangle|,\sigma\langle 1 \rangle)}(z_1), S_{(|\sigma\langle 2 \rangle|,\sigma\langle 2 \rangle)}(z_1), \ldots, S_{(|\sigma\langle 4l \rangle|,\sigma\langle 4l \rangle)}(z_1) \in B$ for $\sigma\langle j \rangle \in \Gamma_k$.

For a fixed $j \in \{1, 2, \ldots, 4l\}$, if

$$S_{(|\sigma\langle j1 \rangle|,\sigma\langle j1 \rangle)}(z_1) = S_{(|\sigma\langle j2 \rangle|,\sigma\langle j2 \rangle)}(z_1) = \cdots = S_{(|\sigma\langle j \rangle|,\sigma\langle j \rangle)}(z_1) \tag{4.2}$$

for $\sigma\langle ji \rangle \in \Gamma_k$, we get

$$S_{(|\sigma\langle ji \rangle|,\sigma\langle ji \rangle)}(x) = S_{(|\sigma\langle j \rangle|,\sigma\langle j \rangle)}(x), \quad \forall x \in [0,1] \ i = 1, 2, \ldots. \tag{4.3}$$

By induction, we can get that

$$N(k) \leq \left[\frac{k+1}{2}\right] \leq k \tag{4.4}$$

for $j = 1, \ldots, 4l$, where $N(k)$ is the cardinality of i which satisfies (4.3).

Thus, for every $\epsilon > 0$, let $N_0 \hat{=} \min\{k : k \leq r^{-k\epsilon}\}$, take $\delta \hat{=} \min\{r^{N_0}, r\}$, we have $\mathrm{Card}\, H_M \leq (4l)^2 (\mathrm{diam}(M))^{-\epsilon}$. It is easy to show that $\dim K = \mathrm{Dim}\, K = 1$. $\square$

EXAMPLE 4.3. Let $(\Omega, \mathcal{F}, P)$ be $((0,1], \mathcal{B}(0,1], \mathcal{L})$. Take $J = [0,1]$, then we will construct a random recursive set, for $\omega \in \Omega$, by the following procedure:

(I) The first three steps:

(1) let

$$S_1^1(x) = \frac{x}{3}, \qquad S_2^1(x) = \frac{x}{3} + \frac{2}{3}; \tag{4.5}$$

(2) let

$$S_{11}^1(x) = S_1^1 \circ S_1^1(x) = \frac{x}{9}, \qquad S_{12}^1(x) = \frac{x}{3^2} + \left(\left(\frac{1}{3} \right)^2 - \left(\frac{1}{3} \right)^3 \omega \right),$$

$$S_{21}^1 = S_2^1 \circ S_1^1(x), \qquad S_{22}^1 = S_2^1 \circ S_2^1(x); \tag{4.6}$$

(3) let

$$S_{111}^1(x) = S_1^1 \circ S_1^1 \circ S_1^1(x), \qquad S_{112}^1(x) = S_1^1 \circ S_1^1 \circ S_2^1(x) = \frac{x}{27} + \frac{2}{27},$$

$$S_{121}^1(x) = S_{112}^1 + \left(\frac{1}{3} \right)^3, \qquad S_{122}^1(x) = S_{112}^1 + \left(\left(\frac{1}{3} \right)^2 - \left(\frac{1}{3} \right)^3 \omega \right),$$

$$S_{211}^1(x) = S_2^1 \circ S_1^1 \circ S_1^1(x), \qquad S_{212}^1(x) = S_2^1 \circ S_1^1 \circ S_2^1(x),$$

$$S_{221}^1(x) = S_2^1 \circ S_2^1 \circ S_1^1(x), \qquad S_{222}^1(x) = S_2^1 \circ S_2^1 \circ S_2^1(x). \tag{4.7}$$

(II) The second three steps. Replacing $[0,1]$ by $[0,1/27]$, we repeat the above iterative procedure:

(1) let

$$S_1^2(x) = \frac{x/27}{3}, \qquad S_2^2(x) = \frac{x/27}{3} + \frac{2/27}{3}; \tag{4.8}$$

(2) let

$$S_{11}^2(x) = S_1^2 \circ S_1^2(x), \qquad S_{12}^2(x) = \frac{x/27}{3^2} + \left(\left(\frac{1}{3} \right)^{2+3} - \left(\frac{1}{3} \right)^{3+3} \omega \right),$$

$$S_{21}^2(x) = S_2^2 \circ S_1^2(x), \qquad S_{22}^2(x) = S_2^2 \circ S_2^2(x); \tag{4.9}$$

(3) let

$$S_{111}^2(x) = S_1^2 \circ S_1^2 \circ S_1^2(x), \qquad S_{112}^2(x) = S_1^2 \circ S_1^2 \circ S_2^2(x) = \frac{x/27}{27} + \frac{2/27}{27},$$

$$S_{121}^2(x) = S_{112}^2 + \left(\frac{1}{3} \right)^{3+3}, \qquad S_{122}^2(x) = S_{112}^2 + \left(\left(\frac{1}{3} \right)^{2+3} - \left(\frac{1}{3} \right)^{3+3} \omega \right),$$

$$S_{211}^2(x) = S_2^2 \circ S_1^2 \circ S_1^2(x), \qquad S_{212}^2(x) = S_2^2 \circ S_1^2 \circ S_2^2(x),$$

$$S_{221}^2(x) = S_2^2 \circ S_2^2 \circ S_1^2(x), \qquad S_{222}^2(x) = S_2^2 \circ S_2^2 \circ S_2^2(x). \tag{4.10}$$

(III) Replacing $[0,1/27]$ by $[0,1/27 \times 27]$, we repeat the same iterative procedure again. And so on.

It is not easy to find an open set O such that $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \sigma \in D\}$ satisfies the OSC, but we can easily see that for every $\omega \in \Omega$, $\{(S_{\sigma*1},\ldots,S_{\sigma*N}), \sigma \in D\}$ satisfies the FIP, so according to Theorem 1.1, we obtain that $\mathrm{Dim}\, K = \dim K = \ln 2 / \ln 3$ a.e. and the Hausdorff measure of the random set $K(\omega)$ is finite and positive a.e.

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