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ON NEW GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITIES

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We give some new generalizations of Hardy's integral inequalities.

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1. Introduction. The classical Hardy inequality [3] states that: for $f(x) \geq 0$, $p > 1$, $1/p + 1/q = 1$, and $0 < \int_0^\infty f^p(x)\,dx < \infty$,

$$\int_0^\infty \left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx < q^p \int_0^\infty f^p(t)\,dt, \tag{1.1}$$

where $q = p/(p-1)$ is the best possible constant.

The dual form of (1.1) is as follows: if $0 < \int_0^\infty (xf(x))^p\,dx < \infty$, then

$$\int_0^\infty \left(\int_x^\infty f(t)\,dt\right)^p dx < p^p \int_0^\infty (tf(t))^p\,dt, \tag{1.2}$$

where the constant p^p in (1.2) is still best possible.

Bicheng et al. [2] gave some new generalizations of (1.1) which can be stated as follows:

$$\int_a^b \left(\frac{1}{x}\int_a^x f(t)\,dt\right)^p dx < q^p \left[1 - \left(\frac{a}{b}\right)^{1/q}\right]^p \int_a^b f^p(t)\,dt; \tag{1.3}$$

$$\int_a^\infty \left(\frac{1}{x}\int_a^x f(t)\,dt\right)^p dx < q^p \int_a^\infty \left[1-\theta_p(t)\right] f^p(t)\,dt \quad (0 < \theta_p(t) < 1); \tag{1.4}$$

$$\int_0^b \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx < q^p \int_0^b \left[1 - \left(\frac{t}{b}\right)^{1/q}\right] f^p(t)\,dt, \tag{1.5}$$

where $\theta_p(t) = (1/p)\sum_{k=1}^\infty \binom{p}{k+1}(-1)^{k-1}(a/t)^{k/q} > 0$ for $t > a > 0$, and $\theta_p(a) = 1/q$.

Recently, Becheng and Debnath [1] gave improvement of (1.3) and some generalizations of (1.2):

$$\begin{gathered}
\int_a^b \left(\frac{1}{x}\int_a^x f(t)\,dt\right)^p dx < q^p \eta_p(a,b) \int_a^b f^p(t)\,dt; \\
\int_a^\infty \left(\int_x^\infty f(t)\,dt\right)^p dx < p^p \int_a^\infty \left[1 - \left(\frac{a}{t}\right)^{1/p}\right] (tf(t))^p\,dt; \\
\int_0^b \left(\int_x^b f(t)\,dt\right)^p dx < p^p \int_0^b \mu_p(t) (tf(t))^p\,dt,
\end{gathered} \tag{1.6}$$

where the constants $\eta_p(a,b) = \max_{a\le t\le b}\{(1/q)t^{1/q}\int_t^b x^{-1-1/q}[1-(a/x)^{1/q}]^{p-1}dx\}$, $\mu_p(t) = (1/p)\{1-(t/b)^{1/p}\}^p(b/t)^{1/p}$.

In this paper, we show some new improvements and generalizations of the inequalities (1.1) and (1.2).

2. Main results

Lemma 2.1. *Let $a \ge 0$, $p > 1$, $1/p+1/q = 1-1/r$, $f \ge 0$, $r > 1$, and $0 < \int_a^\infty f^p(t)dt < \infty$. Then, there exists a real number $x_0 \in (a,\infty)$ such that, for any $x > x_0$, the following inequality is true:*

$$\left(\int_a^x f(t)dt\right)^p < \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1}\left(1-\frac{1}{r}\right)^{p-1} \times\left(x^{1-1/(1-1/r)q(p-1)}-a^{1-1/(1-1/r)q(p-1)}\right)^{p-1}\int_a^x t^{1/(1-1/r)q}f^p(t)dt. \tag{2.1}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}\left(\int_a^x f(t)dt\right)^p &= \left(\int_a^x t^{1/(1-1/r)pq}f(t)t^{-1/(1-1/r)pq}dt\right)^p\\ &\le \int_a^x t^{1/(1-1/r)q}f^p(t)dt\left(\int_a^x\left(t^{-1/(1-1/r)pq}\right)^{p/(p-1)}dt\right)^{p-1}\\ &= \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1}\left(1-\frac{1}{r}\right)^{p-1}\\ &\quad\times\left(x^{1-1/(1-1/r)q(p-1)}-a^{1-1/(1-1/r)q(p-1)}\right)^{p-1}\int_a^x t^{1/(1-1/r)q}f^p(t)dt.\end{aligned} \tag{2.2}$$

We need to show that there exists a real number $x_0 \in (a,\infty)$, such that (2.2) does not assume equality for any $x > x_0$. Otherwise, there exists $x = x_n \in (a,\infty)$, where $n = 1,2,3,\dots,x_n \uparrow \infty$, such that (2.2) becomes an equality. By the same argument, there exists a real number $c > 0$, and an integer N, such that for $n > N$,

$$\left(t^{1/(1-1/r)pq}f(t)\right)^p = c\left(t^{-1/(1-1/r)pq}\right)^{p/(p-1)} \quad \text{a.e. in } [a,x_n]. \tag{2.3}$$

Hence

$$\begin{aligned}\int_a^{x_n} f^p(t)dt &= \int_a^{x_n} c\frac{t^{-1/(1-1/r)q(p-1)}}{t^{1/(1-1/r)q}}dt\\ &= \int_a^{x_n} ct^{-p/(1-1/r)q(p-1)}dt \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{2.4}$$

This is a contradiction to the fact that $0 < \int_a^\infty f^p(t)dt < \infty$. Hence, (2.1) holds true and the proof is complete. ◻

Lemma 2.2. *Let $b > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and let $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t)dt < \infty$. Then, there exists a real number $x_0 \in (0,b)$ such that, for any $x \in (0,x_0)$, the following inequality is true:*

$$\left(\int_x^b f(t)dt\right)^p < \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\left(x^{-1/(1-1/r)p} - b^{-1/(1-1/r)p}\right)^{p-1} \times \int_x^b t^{p-1+(p-1)/(1-1/r)p} f^p(t)dt. \tag{2.5}$$

Proof. For any $x \in (0,b)$, by Hölder's inequality, we have

$$\begin{aligned}\left(\int_x^b f(t)dt\right)^p &= \left[\int_x^b t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t) t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2} dt\right]^p \\ &\leq \int_x^b t^{(1+(1-1/r)p)(p-1)/(1-1/r)p} f^p(t)dt \left(\int_x^b t^{-(1+(1-1/r)p)/(1-1/r)p} dt\right)^{p-1} \\ &= \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\left(x^{-1/(1-1/r)p} - b^{-1/(1-1/r)p}\right)^{p-1} \\ &\quad \times \int_x^b t^{p-1+(p-1)/(1-1/r)p} f^p(t)dt.\end{aligned} \tag{2.6}$$

We need to show that there exists a real number $x_0 \in (0,b)$, such that (2.6) does not assume equality for any $x \in (0,x_0)$. Otherwise, there exists $x = x_n \in (0,b)$, where $n = 1,2,3,\ldots, x_n \downarrow 0$, such that (2.6) becomes an equality. Then there exist c_n and d_n which are not always zero, such that (see [4, page 29])

$$\begin{aligned}&c_n\left[t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t)\right]^p \\ &\quad = d_n\left[t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2}\right]^{p/(p-1)} \quad \text{a.e. in } [x_n,b].\end{aligned} \tag{2.7}$$

Since $f(t) \neq 0$ a.e. in $(0,b)$, there exists an integer N such that, for $n > N$, $f(t) \neq 0$ a.e. in $(0,x_n)$. Thus, for both $c_n = c \neq 0$ and $d_n = d \neq 0$ for $n > N$,

$$\int_0^b t^{p-1+1/(1-1/r)} f^p(t)dt = \lim_{n\to\infty}\int_{x_n}^b \frac{t^{-(1+1/(1-1/r)p)}}{t^{1-(1+1/(1-1/r)p)}}dt = \frac{d}{c}\lim_{n\to\infty}\int_{x_n}^b \frac{dt}{t} = \infty. \tag{2.8}$$

This contradicts the fact that $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t)dt < \infty$. Hence, (2.5) is valid and this completes the proof of the lemma. □

Lemma 2.3. *Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty t^{p-1+1/(1-1/r)} f^p(t)dt < \infty$. Then, there exists a real number $x_0 \in (a,\infty)$ such that, for any $x \in (a,x_0)$, the following inequality is true:*

$$\left(\int_x^\infty f(t)dt\right)^p < \left(\left(1-\frac{1}{r}\right)p\right)^{p-1} x^{-(p-1)/(1-1/r)p} \int_x^\infty t^{p-1+(p-1)/(1-1/r)p} f^p(t)dt. \tag{2.9}$$

Proof. For any $x \in (a,\infty)$, by Hölder's inequality, we have

$$\left(\int_x^\infty f(t)dt\right)^p \le \left(\left(1-\frac{1}{r}\right)p\right)^{p-1} x^{-(p-1)/(1-1/r)p} \int_x^\infty t^{p-1+(p-1)/(1-1/r)p} f^p(t)dt. \tag{2.10}$$

We show that there exists a real number $x_0 \in (a,\infty)$, such that (2.10) does not assume equality for any $x \in (a,x_0)$. Otherwise, there exists $x = x_n \in (a,\infty)$, where $n = 1,2,3,\dots,x_n \downarrow a$, such that (2.10) becomes an equality. By the same argument there exist a real number $c > 0$, and an integer N, such that for $n > N$,

$$\begin{aligned}&\left[t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t)\right]^p \\ &\quad = c\left[t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2}\right]^{p/(p-1)} \quad \text{a.e. in } [x_n,\infty),\end{aligned} \tag{2.11}$$

and hence $\int_a^\infty t^{p-1+1/(1-1/r)} f^p(t)dt = c\lim_{n\to\infty}\int_{x_n}^\infty (dt/t) = \infty$. This contradicts the fact that $0 < \int_a^\infty t^{p-1+1/(1-1/r)} f^p(t)dt < \infty$. Hence (2.9) is valid and this completes the proof of the lemma. □

Theorem 2.4. *Let $0 < a < b$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \ge 0$, $r > 1$, and $0 < \int_a^\infty f^p(t)dt < \infty$. Then*

$$\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx < \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^p \left(1-\frac{1}{r}\right)^p \eta(a,b)\int_a^b f^p(t)dt, \tag{2.12}$$

where the constant

$$\begin{aligned}\eta(a,b) &= \max_{a\le t\le b}\left\{\frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q}\right. \\ &\qquad\left.\times\int_t^b x^{-1-1/(1-1/r)q}\left[1-\left(\frac{a}{x}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1} dx\right\}, \\ \eta(a,b) &< \frac{(p+q)(p-1)-p}{p(p-1)}\left[1-\left(\frac{a}{b}\right)^{1-1/(1-1/r)q(p-1)}\right]^p.\end{aligned} \tag{2.13}$$

Proof. In view of the proof of Lemma 2.1, we obtain

$$\begin{aligned}&\int_a^b \left(\frac{1}{x}\int_a^x f(t)dt\right)^p dx \\ &\quad < \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^{p-1}\left(1-\frac{1}{r}\right)^{p-1} \\ &\qquad \times \int_a^b \left\{\int_t^b x^{-1-1/(1-1/r)q}\left[1-\left(\frac{a}{x}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1} dx\right\} t^{1/(1-1/r)q} f^p(t)dt \\ &\quad = \left(\frac{pq(p-1)}{(p+q)(p-1)-p}\right)^p \left(1-\frac{1}{r}\right)^p \int_a^b g(t) f^p(t)dt,\end{aligned} \tag{2.14}$$

where the weight function $g(t)$ is defined by

$$g(t) := \frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q} \times \int_t^b x^{-1-1/(1-1/r)q}\left[1-\left(\frac{a}{x}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1} dx, \quad t \in [a,b]. \tag{2.15}$$

Setting $\eta(a,b) := \max_{a \le t \le b} g(t)$, since $g(t)$ is a nonconstant continuous function, then by (2.14) we have (2.12). Since $g(b) = 0$, and for any $t \in [a,b)$,

$$\begin{aligned} g(t) &< \frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q} \int_t^b x^{-1-1/(1-1/r)q}\left[1-\left(\frac{a}{b}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1} dx \\ &= \frac{(p+q)(p-1)-p}{pq(p-1)}\left[1-\left(\frac{a}{b}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1}\left[1-\left(\frac{t}{b}\right)^{1/(1-1/r)q}\right] \\ &\le \frac{(p+q)(p-1)-p}{pq(p-1)}\left[1-\left(\frac{a}{b}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p-1}\left[1-\left(\frac{a}{b}\right)^{1/(1-1/r)q}\right] \\ &< \frac{(p+q)(p-1)-p}{pq(p-1)}\left[1-\left(\frac{a}{b}\right)^{1-1/(1-1/r)q(p-1)}\right]^{p}. \end{aligned} \tag{2.16}$$

This completes the proof. □

Theorem 2.5. *Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \ge 0$, $r > 1$, and $0 < \int_a^\infty (tf(t))^p dt < \infty$, $0 < \int_a^\infty t^{p-1+1/(1+1/r)} f^p(t) dt < \infty$. Then*

$$\int_a^\infty \left(\int_x^\infty f(t)dt\right)^p dx < \left(\left(1-\frac{1}{r}\right)p\right)^p \frac{r}{r-p}\int_a^\infty \left[1-\left(\frac{a}{t}\right)^{(r-p)/(r-1)p}\right](tf(t))^p dt. \tag{2.17}$$

Proof. Applying (2.9), we have

$$\begin{aligned} &\int_a^\infty \left(\int_x^\infty f(t)dt\right)^p dx \\ &\quad < \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\int_a^\infty x^{-(p-1)/(1-1/r)p}\int_x^\infty t^{p-1+*(p-1)/(1-1/r)p} f^p(t)dt\,dx \\ &\quad = \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\int_a^\infty \left(\int_a^t x^{-(p-1)/(1-1/r)p} dx\right) t^{p-1+(p-1)/(1-1/r)p} f^p(t)dt \\ &\quad = \left(\left(1-\frac{1}{r}\right)p\right)^{p}\frac{r}{r-p}\int_a^\infty \left[1-\left(\frac{a}{t}\right)^{(r-p)/(r-1)p}\right](tf(t))^p dt. \end{aligned} \tag{2.18}$$

Hence, (2.17) is valid. This completes the proof of the theorem. □

Theorem 2.6. *Let $b > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \ge 0$, $r > 1$, and $0 < \int_0^b (tf(t))^p dt < \infty$, $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t) dt < \infty$. Then*

$$\int_0^b \left(\int_x^b f(t)dt\right)^p dx < \left(\left(1-\frac{1}{r}\right)p\right)^p \int_0^b \mu(t)(tf(t))^p dt, \tag{2.19}$$

where $\mu(t) := 1/(1-1/r)p\{\int_0^t x^{-(p-1)/(1-1/r)p}[1-(x/b)^{1/(1-1/r)p}]^{p-1}dx\}t^{(p-r)/(r-1)p}$, $t \in (0,b]$.

Proof. Applying (2.5), we have

$$\begin{aligned}\int_0^b\left(\int_x^b f(t)dt\right)^p dx &< \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\int_0^b\left(x^{-1/(1-1/r)p}-b^{-1/(1-1/r)p}\right)^{p-1}\\ &\quad\times\int_x^b t^{p-1+(p-1)/(1-1/r)p}f^p(t)\,dt\,dx\\ &= \left(\left(1-\frac{1}{r}\right)p\right)^{p-1}\int_0^b\left(\int_0^t x^{-(p-1)(1-1/r)p}\left[1-\left(\frac{x}{b}\right)^{1/(1-1/r)p}\right]^{p-1}dx\right)\\ &\quad\times t^{(p-r)/(r-1)p}(tf(t))^p dt\\ &= \left(\left(1-\frac{1}{r}\right)p\right)^p\int_0^b \mu(t)(tf(t))^p dt,\end{aligned} \tag{2.20}$$

where $\mu(t) := 1/(1-1/r)p\{\int_0^t x^{-(p-1)/(1-1/r)p}[1-(x/b)^{1/(1-1/r)p}]^{p-1}dx\}t^{(p-r)/(r-1)p}$, $t \in (0,b]$. This proves (2.19) and the proof of the theorem is complete. □

Remark 2.7. Let $r \to \infty$, (2.1) changes into [2, (2.3)]. Hence, (2.1) is a generalization of [2, (2.3)].

Remark 2.8. Let $r \to \infty$, (2.5) and (2.9) change into [1, (3.1) and (3.5)], respectively. Hence (2.5) and (2.9) is generalization of [1, (3.1) and (3.5)], respectively.

Remark 2.9. Let $r \to \infty$, (2.12), (2.17), and (2.19) change into [1, (3), (4), and (5)], respectively. Hence, (2.12), (2.17), and (2.19) is generalization of [1, (3), (4), and (5)], respectively.

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