NULL DISTRIBUTION OF MULTIPLE CORRELATION COEFFICIENT UNDER MIXTURE NORMAL MODEL

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The multiple correlation coefficient is used in a large variety of statistical tests and regression problems. In this article, we derive the null distribution of the square of the sample multiple correlation coefficient, R^2, when a sample is drawn from a mixture of two multivariate Gaussian populations. The moments of $1 - R^2$ and inverse Mellin transform have been used to derive the density of R^2.

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1. Introduction. Suppose that $\mathbf{x}(p \times 1)$, $\boldsymbol{\mu}(p \times 1)$, and $\Sigma(p \times p) > 0$ are partitioned as $\mathbf{x} = \begin{pmatrix} x_1 \\ \mathbf{x}^{(2)} \end{pmatrix}$, $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \boldsymbol{\mu}^{(2)} \end{pmatrix}$, and $\Sigma = \begin{pmatrix} \sigma_{11} & \boldsymbol{\sigma}'_{21} \\ \boldsymbol{\sigma}_{21} & \Sigma_{22} \end{pmatrix}$, where $\mathbf{x}^{(2)} = (x_2, \ldots, x_p)'$ and $\boldsymbol{\mu}^{(2)} = (\mu_2, \ldots, \mu_p)'$ are $(p-1) \times 1$ and Σ_{22} is $(p-1) \times (p-1)$, so that $\mathrm{Var}(x_1) = \sigma_{11}$, $\mathrm{Cov}(\mathbf{x}^{(2)}) = \Sigma_{22}$, and $\boldsymbol{\sigma}_{12}$ is the $(p-1) \times 1$ vector of covariances between x_1 and $x_2, \ldots, x_p$. The multiple correlation coefficient between x_1 and $\mathbf{x}^{(2)}$, denoted by $\bar{R}_{1 \cdot 2 \cdots p}$, is defined as

$$\bar{R}_{1 \cdot 2 \cdots p} = \left(\frac{\boldsymbol{\sigma}'_{21} \Sigma_{22}^{-1} \boldsymbol{\sigma}_{21}}{\sigma_{11}} \right)^{1/2}. \tag{1.1}$$

Let A be the sample sum of squares and products matrix formed from N independent observations on $\mathbf{x}$. Partition A as $A = \begin{pmatrix} a_{11} & \mathbf{a}'_{21} \\ \mathbf{a}_{21} & A_{22} \end{pmatrix}$, where A_{22} is $(p-1) \times (p-1)$. The sample multiple correlation coefficient between x_1 and $\mathbf{x}^{(2)}$ is defined by

$$R = \left(\frac{\mathbf{a}'_{21} A_{22}^{-1} \mathbf{a}_{21}}{a_{11}} \right)^{1/2}. \tag{1.2}$$

It is well known that, when the underlying population is normal, the random matrix A has Wishart distribution with $n = N - 1$ degrees of freedom and parameter matrix Σ. Further, $\bar{R}_{1 \cdot 2 \cdots p} = 0$ if and only if x_1 is independent of $\mathbf{x}^{(2)} = (x_2, \ldots, x_p)$. Furthermore, when the population multiple correlation coefficient $\bar{R}_{1 \cdot 2 \cdots p}$ is zero, the distribution of R^2 is beta with parameters $(1/2)(p-1)$ and $(1/2)(N-p)$.

In practice, it is often the case that the random variables are not normally distributed. When such is the case, how would the departure from the normality affect the conventional inference procedure? Specifically, one may wonder what would be the sampling distributions of some commonly used statistics? For providing some answers to the above questions, Srivastava and Awan [9] and Tan [11] derived the distribution of the sample sum of squares and products matrix when sampling from a mixture of two multivariate normal distributions. The normal mixture is defined as follows:

$$f(\mathbf{x}) = \epsilon N_p(\boldsymbol{\mu}_1, \Sigma; \mathbf{x}) + (1 - \epsilon) N_p(\boldsymbol{\mu}_2, \Sigma; \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^p, \tag{1.3}$$

where

$$N_p(\boldsymbol{\mu},\Sigma;\mathbf{x})$$
$$= (2\pi)^{-(1/2)p}\det(\Sigma)^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}, \quad \mathbf{x}\in\mathbb{R}^p, \ \boldsymbol{\mu}\in\mathbb{R}^p, \ \Sigma>0,$$
(1.4)

and $1-\epsilon$ is known as the degree of contamination. This model is very common in medical, biological, and agricultural experiments (Titterington et al. [12]). For results on the distribution theory and robustness studies of certain test statistics when sampling from a mixture normal model, see Srivastava [8], Srivastava and Awan [9, 10], Kabe and Gupta [5], Amey and Gupta [2], and Nagar and Castañeda [7].

Srivastava [8], using certain transformations, derived the null distribution of multiple correlation coefficient when sampling from a mixture of two multivariate normal distributions (see also Gupta and Kabe [3]). Amey [1] integrated the joint density of a_{11}, $\mathbf{a}_{21}$, and A_{22} suitably to derive the density of R^2 and studied its robustness.

In this article, we derive the null distribution of R^2 when sampling from a mixture of two multivariate normal distributions. First, we derive the hth null moment of $1-R^2$. Then, by using the inverse Mellin transform, the density of $1-R^2$ is obtained from which the density of R^2 is deduced.

Note that R^2 is a function of the elements of sample sum of squares and products matrix A. Therefore, in our derivation, we use the distribution of A when sampling from the above model. Srivastava and Awan [9] and Tan [11] have shown that the density of A, when sampling from (1.3), is a binomial sum of linear noncentral Wishart densities:

$$f(A) = \sum_{\gamma=0}^{N}\binom{N}{\gamma}\epsilon^\gamma(1-\epsilon)^{N-\gamma}W_p(n,\Sigma,c_\gamma^2\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}';A),$$
(1.5)

where $n=N-1$, $c_\gamma^2=\gamma(N-\gamma)/N$, and $\boldsymbol{\nu}=(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)$. Here $W_p(n,\Sigma,c_\gamma^2\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}';A)$ represents the noncentral Wishart density with n degrees of freedom and noncentrality parameter matrix $c_\gamma^2\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'$ defined by

$$K_p(n,\Sigma,\boldsymbol{\nu})\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}A\right)\det(A)^{(1/2)(n-p-1)}{}_0F_1^{(p)}\left(\frac{1}{2}n;\frac{1}{4}c_\gamma^2\Sigma^{-1}A\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\right),$$
(1.6)

where

$$K_p(n,\Sigma,\boldsymbol{\nu}) = \left\{2^{(1/2)pn}\Gamma_p\left(\frac{1}{2}n\right)\det(\Sigma)^{(1/2)n}\right\}^{-1}\operatorname{etr}\left(-\frac{1}{2}c_\gamma^2\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\right)$$
(1.7)

and $\Gamma_m(a)=\pi^{(1/4)m(m-1)}\prod_{j=1}^{m}\Gamma(a-(1/2)(j-1))$.

2. Null moments of $1-R^2$. In this section, we derive moments of $1-R^2$ when $\bar{R}_{1\cdot 2\cdots p}=0$ (or equivalently $\boldsymbol{\sigma}_{21}=\mathbf{0}$). Let $\Sigma_0=\left(\begin{smallmatrix}\sigma_{11} & \mathbf{0}'\\ \mathbf{0} & \Sigma_{22}\end{smallmatrix}\right)$ and $U=1-R^2$. Since a_{11} is scalar, then

$$U = 1-R^2 = 1-\frac{\mathbf{a}_{21}'A_{22}^{-1}\mathbf{a}_{21}}{a_{11}} = \frac{\det(A)}{a_{11}\det(A_{22})}.$$
(2.1)

The hth null moment of U is given by

$$E(U^h) = \sum_{y=0}^{N} \binom{N}{y} \epsilon^y (1-\epsilon)^{N-y} E_y(U^h), \tag{2.2}$$

$$E_y(U^h) = K_p(n,\Sigma_0,\boldsymbol{v}) \int_{A>0} \text{etr}\left(-\frac{1}{2}\Sigma_0^{-1}A\right) a_{11}^{-h} \det(A_{22})^{-h}$$
$$\times \det(A)^{(1/2)(n-p-1)+h} {}_0F_1^{(p)}\left(\frac{1}{2}n; \frac{1}{4}c_y^2\Sigma_0^{-1}A\Sigma_0^{-1}\boldsymbol{vv}'\right)dA. \tag{2.3}$$

Replacing a_{11}^{-h} and $\det(A_{22})^{-h}$ by their integral representations, namely

$$a_{11}^{-h} = \frac{1}{2^h\Gamma(h)} \int_0^{\infty} \exp\left(-\frac{1}{2}a_{11}y_1\right) y_1^{h-1} dy_1, \quad \text{Re}(h) > 0,$$

$$\det(A_{22})^{-h} = \frac{1}{2^{(p-1)h}\Gamma_{p-1}(h)} \int_{Y_{22}>0} \text{etr}\left(-\frac{1}{2}A_{22}Y_{22}\right) \tag{2.4}$$
$$\times \det(Y_{22})^{h-(1/2)(p-1+1)} dY_{22}, \quad \text{Re}(h) > \frac{1}{2}(p-2),$$

respectively, in (2.3) and integrating A, the moment expression is rewritten as

$$E_y(U^h) = \frac{2^{(1/2)np} K_p(n,\Sigma_0,\boldsymbol{v})\Gamma_p((1/2)n+h)}{\Gamma(h)\Gamma_{p-1}(h)}$$
$$\times \int_0^{\infty} y_1^{h-1} \int_{Y_{22}>0} \det(Y_{22})^{h-(1/2)p} \det(\Sigma_0^{-1}+Y)^{-(1/2)n-h} \tag{2.5}$$
$$\times {}_1F_1^{(p)}\left(\frac{1}{2}n+h; \frac{1}{2}n; \frac{1}{2}c_y^2\Sigma_0^{-1}(\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}'\right)dy_1dY_{22},$$

where $Y = \begin{pmatrix} y_1 & 0 \\ 0 & Y_{22} \end{pmatrix}$ and ${}_1F_1^{(p)}$ is the confluent hypergeometric function of matrix argument (Gupta and Nagar [4]). Since $\text{rank}(\Sigma_0^{-1}(\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}') = 1$, the only nonzero characteristic root of the matrix $\Sigma_0^{-1}(\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}'$ is $\text{tr}((\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}'\Sigma_0^{-1})$ and therefore,

$$\begin{aligned}{}_1F_1^{(p)}&\left(\frac{1}{2}n+h; \frac{1}{2}n; \frac{1}{2}c_y^2\Sigma_0^{-1}(\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}'\right) \\ &= {}_1F_1\left(\frac{1}{2}n+h; \frac{1}{2}n; \frac{1}{2}c_y^2\text{tr}((\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{vv}'\Sigma_0^{-1})\right),\end{aligned} \tag{2.6}$$

where ${}_1F_1$ is the confluent hypergeometric function of scalar argument (see [6]). Substituting (2.6) in (2.5) and expanding ${}_1F_1$ in series form, the moment expression simplifies to

$$E_y(U^h) = \frac{2^{(1/2)np} K_p(n,\Sigma_0,\boldsymbol{v})\Gamma_p((1/2)n+h)}{\Gamma(h)\Gamma_{p-1}(h)} \sum_{t=0}^{\infty} \left(\frac{c_y^2}{2}\right)^t \frac{((1/2)n+h)_t}{((1/2)n)_t t!}$$
$$\times \int_0^{\infty} y_1^{h-1} \int_{Y_{22}>0} \det(Y_{22})^{h-(1/2)p} \det(\Sigma_0^{-1}+Y)^{-(1/2)n-h} \tag{2.7}$$
$$\times [\boldsymbol{v}'\Sigma_0^{-1}(\Sigma_0^{-1}+Y)^{-1}\Sigma_0^{-1}\boldsymbol{v}]^t dy_1dY_{22},$$

where $(a)_r = a(a+1)\cdots(a+r-1)$ and $(a)_0 = 1$. Noting that Σ_0 is a block diagonal matrix, we obtain

$$\left[\boldsymbol{v}'\Sigma_0^{-1}\left(\Sigma_0^{-1}+Y\right)^{-1}\Sigma_0^{-1}\boldsymbol{v}\right]^t$$

$$= \left[v_1^2\sigma_{11}^{-1}\left(1+\sigma_{11}y_1\right)^{-1}+\boldsymbol{v}_2'\Sigma_{22}^{-1}\left(\Sigma_{22}^{-1}+Y_2\right)^{-1}\Sigma_{22}^{-1}\boldsymbol{v}_2\right]^t$$

$$= \sum_{k+\ell=t}\frac{t!}{k!\ell!}\left\{v_1^2\sigma_{11}^{-1}\left(1+\sigma_{11}y_1\right)^{-1}\right\}^k\left\{\boldsymbol{v}_2'\Sigma_{22}^{-1}\left(\Sigma_{22}^{-1}+Y_{22}\right)^{-1}\Sigma_{22}^{-1}\boldsymbol{v}_2\right\}^\ell, \tag{2.8}$$

$$\det\left(\Sigma_0^{-1}+Y\right) = \left(\sigma_{11}\right)^{-1}\left(1+\sigma_{11}y_1\right)\det\left(\Sigma_{22}\right)^{-1}\det\left(I_{p-1}+\Sigma_{22}Y_{22}\right).$$

Now substituting (2.8) in (2.7), we have

$$E_Y(U^h) = \frac{2^{(1/2)np}K_p(n,\Sigma_0,\boldsymbol{v})\Gamma_p((1/2)n+h)}{\det(\Sigma_0)^{(1/2)n+h}\Gamma(h)\Gamma_{p-1}(h)}$$

$$\times\sum_{t=0}^{\infty}\left(\frac{c_Y^2}{2}\right)^t\frac{((1/2)n+h)_t}{((1/2)n)_t}\sum_{k+\ell=t}\frac{1}{k!\ell!}\left(\frac{v_1^2}{\sigma_{11}}\right)^k$$

$$\times\int_0^{\infty}y_1^{h-1}\left(1+\sigma_{11}y_1\right)^{-((1/2)n+h+k)}dy_1 \tag{2.9}$$

$$\times\int_{Y_{22}>0}\det\left(Y_{22}\right)^{h-(1/2)p}\det\left(I_{p-1}+\Sigma_{22}Y_{22}\right)^{-(1/2)n-h}$$

$$\times\left\{\boldsymbol{v}_2'\Sigma_{22}^{-1}\left(\Sigma_{22}^{-1}+Y_{22}\right)^{-1}\Sigma_{22}^{-1}\boldsymbol{v}_2\right\}^\ell dY_{22}.$$

Substituting $Z = (I_{p-1}+\Sigma_{22}^{1/2}Y_{22}\Sigma_{22}^{1/2})^{-1}$, the integral involving Y_{22} is evaluated as

$$\int_{Y_{22}>0}\det\left(Y_{22}\right)^{h-(1/2)p}\det\left(I_{p-1}+\Sigma_{22}Y_{22}\right)^{-(1/2)n-h}\left\{\boldsymbol{v}_2'\Sigma_{22}^{-1}\left(\Sigma_{22}^{-1}+Y_{22}\right)^{-1}\Sigma_{22}^{-1}\boldsymbol{v}_2\right\}^\ell dY_{22}$$

$$= \det\left(\Sigma_{22}\right)^{-h}\int_{0<Z<I_{p-1}}\det(Z)^{(1/2)(n-p)}$$

$$\times\det\left(I_{p-1}-Z\right)^{h-(1/2)p}\left(\boldsymbol{v}_2'\Sigma_{22}^{-1/2}Z\Sigma_{22}^{-1/2}\boldsymbol{v}_2\right)^\ell dZ$$

$$= \det\left(\Sigma_{22}\right)^{-h}\left(\frac{\partial}{\partial\eta}\right)^\ell_{\eta=0}\int_{0<Z<I_{p-1}}\det(Z)^{(1/2)(n-p)}\det\left(I_{p-1}-Z\right)^{h-(1/2)p}$$

$$\times\operatorname{etr}\left(\eta\boldsymbol{v}_2'\Sigma_{22}^{-1/2}Z\Sigma_{22}^{-1/2}\boldsymbol{v}_2\right)dZ$$

$$= \det\left(\Sigma_{22}\right)^{-h}\frac{\Gamma_{p-1}((1/2)n)\Gamma_{p-1}(h)}{\Gamma_{p-1}((1/2)n+h)}\left(\frac{\partial}{\partial\eta}\right)^\ell_{\eta=0}{}_1F_1^{(p-1)}\left(\frac{1}{2}n;\frac{1}{2}n+h;\eta\Sigma_{22}^{-1}\boldsymbol{v}_2\boldsymbol{v}_2'\right)$$

$$= \det\left(\Sigma_{22}\right)^{-h}\frac{\Gamma_{p-1}((1/2)n)\Gamma_{p-1}(h)}{\Gamma_{p-1}((1/2)n+h)}\frac{((1/2)n)_\ell}{((1/2)n+h)_\ell}\left(\boldsymbol{v}_2'\Sigma_{22}^{-1}\boldsymbol{v}_2\right)^\ell, \tag{2.10}$$

where ${}_1F_1^{(p-1)}$ is the confluent hypergeometric function of matrix argument (see [4]).

Collecting terms containing y_1 and integrating, we obtain

$$\int_0^\infty y_1^{h-1}(1+\sigma_{11}y_1)^{-((1/2)n+h+k)}dy_1 = \sigma_{11}^{-h}\frac{\Gamma((1/2)n)\Gamma(h)}{\Gamma((1/2)n+h)}\frac{((1/2)n)_k}{((1/2)n+h)_k}. \qquad (2.11)$$

Substituting (2.10), (2.11), and (1.7) in (2.9) and simplifying the resulting expression using results on gamma function, we get

$$E_y(U^h) = \exp\left(-\frac{1}{2}c_y^2\boldsymbol{v}'\Sigma_0^{-1}\boldsymbol{v}\right)\frac{\Gamma((1/2)n)}{\Gamma[(1/2)(n-p+1)]}\sum_{t=0}^\infty\sum_{\ell+k=t}\left(\frac{c_y^2}{2}\right)^t\frac{((1/2)n)_k((1/2)n)_\ell}{((1/2)n)_t}$$

$$\times\frac{(v_1^2/\sigma_{11})^k(\boldsymbol{v}_2'\Sigma_{22}^{-1}\boldsymbol{v}_2)^\ell}{k!\ell!}\frac{\Gamma[(1/2)(n-p+1)+h]\Gamma((1/2)n+t+h)}{\Gamma((1/2)n+k+h)\Gamma((1/2)n+\ell+h)}$$

$$= \exp\left(-\frac{1}{2}c_y^2\boldsymbol{v}'\Sigma_0^{-1}\boldsymbol{v}\right)\frac{\Gamma((1/2)n)\Gamma[(1/2)(n-p+1)+h]}{\Gamma((1/2)n+h)\Gamma[(1/2)(n-p+1)]}\sum_{k=0}^\infty\left(\frac{c_y^2}{2}\right)^k$$

$$\times\frac{(v_1^2/\sigma_{11})^k}{k!}{}_2F_2\left(\frac{1}{2}n+h+k,\frac{1}{2}n;\frac{1}{2}n+k,\frac{1}{2}n+h;\frac{1}{2}c_y^2\boldsymbol{v}_2'\Sigma_{22}^{-1}\boldsymbol{v}_2\right),$$

$$(2.12)$$

where ${}_2F_2$ is the generalized hypergeometric function of scalar argument (see [6]).

3. Distribution of R^2 under mixture normal model. The density function $f(u)$ of $U = 1-R^2$ is obtained by taking the inverse Mellin transform of $E(U^h)$ as

$$f(u) = \sum_{y=0}^N\binom{N}{y}\epsilon^y(1-\epsilon)^{N-y}f_y(u) \qquad (3.1)$$

with

$$f_y(u) = (2\pi\iota)^{-1}\int_C E_y(U^h)u^{-h-1}dh, \quad 0 < u < 1, \qquad (3.2)$$

where $\iota = \sqrt{-1}$ and C is a suitable contour. Substituting (2.12) in (3.2), we obtain

$$f_y(u) = \exp\left(-\frac{1}{2}c_y^2\boldsymbol{v}'\Sigma_0^{-1}\boldsymbol{v}\right)\frac{\Gamma((1/2)n)}{\Gamma[(1/2)(p-1)]\Gamma[(1/2)(n-p+1)]}\sum_{t=0}^\infty\sum_{k+\ell=t}\left(\frac{c_y^2}{2}\right)^t$$

$$\times\frac{((1/2)n)_k((1/2)n)_\ell}{((1/2)n)_t}\frac{(v_1^2/\sigma_{11})^\ell(\boldsymbol{v}_2'\Sigma_{22}^{-1}\boldsymbol{v}_2)^k}{k!\ell!}u^{(1/2)n+k+\ell-1}(1-u)^{(1/2)(p-3)}$$

$$\times{}_2F_1\left(\frac{1}{2}(p-1)+k,\frac{1}{2}(p-1)+\ell;\frac{1}{2}(p-1);1-u\right), \quad 0 < u < 1,$$

$$(3.3)$$

where ${}_2F_1$ is the Gauss hypergeometric function (see [6]). To obtain (3.3) we have used the result

$$\int_0^1 u^{(1/2)n+h+k+\ell-1}(1-u)^{(1/2)(p-3)}{}_2F_1\left(\frac{1}{2}(p-1)+k,\frac{1}{2}(p-1)+\ell;\frac{1}{2}(p-1);1-u\right)du$$

$$= \frac{\Gamma[(1/2)(p-1)]\Gamma[(1/2)(n-p+1)+h]\Gamma((1/2)n+t+h)}{\Gamma((1/2)n+k+h)\Gamma((1/2)n+\ell+h)}.$$

$$(3.4)$$

The density of $R^2 = 1 - U$ is now derived from the density of U as

$$g(R^2) = \sum_{\gamma=0}^{N} \binom{N}{\gamma} \epsilon^\gamma (1-\epsilon)^{N-\gamma} g_\gamma(R^2), \tag{3.5}$$

where

$$g_\gamma(R^2) = \exp\left(-\frac{1}{2} c_\gamma^2 \boldsymbol{v}' \Sigma_0^{-1} \boldsymbol{v}\right) \frac{\Gamma((1/2)n)}{\Gamma[(1/2)(p-1)]\Gamma[(1/2)(n-p+1)]} \sum_{t=0}^{\infty} \sum_{k+\ell=t} \left(\frac{c_\gamma^2}{2}\right)^t$$

$$\times \frac{((1/2)n)_k((1/2)n)_\ell}{((1/2)n)_t} \frac{(v_1^2/\sigma_{11})^\ell (\boldsymbol{v}_2' \Sigma_{22}^{-1} \boldsymbol{v}_2)^k}{k!\ell!}$$

$$\times (1-R^2)^{(1/2)n+k+\ell-1} (R^2)^{(1/2)(p-3)}$$

$$\times {}_2F_1\left(\frac{1}{2}(p-1)+k, \frac{1}{2}(p-1)+\ell; \frac{1}{2}(p-1); R^2\right), \quad 0 < R^2 < 1. \tag{3.6}$$

By using the result ${}_2F_1(a,b;c;z) = (1-z)^{c-a-b} {}_2F_1(c-a,c-b;c;z)$, the above density can be rewritten as

$$g_\gamma(R^2) = \exp\left(-\frac{1}{2} c_\gamma^2 \boldsymbol{v}' \Sigma_0^{-1} \boldsymbol{v}\right) \frac{\Gamma((1/2)n)}{\Gamma[(1/2)(p-1)]\Gamma[(1/2)(n-p+1)]}$$

$$\times (R^2)^{(1/2)(p-3)} (1-R^2)^{(1/2)(n-p-1)} \sum_{t=0}^{\infty} \sum_{k+\ell=t} \left(\frac{c_\gamma^2}{2}\right)^t \frac{((1/2)n)_k((1/2)n)_\ell}{((1/2)n)_t}$$

$$\times \frac{(v_1^2/\sigma_{11})^\ell (\boldsymbol{v}_2' \Sigma_{22}^{-1} \boldsymbol{v}_2)^k}{k!\ell!} {}_2F_1\left(-k, -\ell; \frac{1}{2}(p-1); R^2\right), \quad 0 < R^2 < 1. \tag{3.7}$$

It is interesting to note that if $\boldsymbol{v} = \boldsymbol{0}$, then the density $g(R^2)$ reduces to

$$g(R^2) = \frac{\Gamma((1/2)n)}{\Gamma[(1/2)(p-1)]\Gamma[(1/2)(n-p+1)]}$$

$$\times (R^2)^{(1/2)(p-3)} (1-R^2)^{(1/2)(n-p-1)}, \quad 0 < R^2 < 1. \tag{3.8}$$

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