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ON THE GEOMETRY OF FREE LOOP SPACES

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We verify the following three basic results on the free loop space LM. (1) We show that the set of all points, where the fundamental form on LM is nondegenerate, is an open subset. (2) The connections of a Fréchet bundle over LM can be extended to S^1-central extensions and, in particular, there exist natural connections on the string structures. (3) The notion of Christoffel symbols and the curvature are introduced on LM and they are described in terms of Christoffel symbols of M.

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1. Introduction. In this paper, we consider the infinite-dimensional Fréchet manifold LM, the free loop space on M, which is the space of all smooth maps from the circle S^1 to a finite-dimensional manifold M. We discuss three different topics as a contribution to the general knowledge on the geometry of loop spaces.

If M is a finite-dimensional Riemannian manifold, Atiyah [1] indicates that LM has a fundamental closed 2-form ω which, unlike the finite-dimensional case, can be degenerate at certain points. A point in LM is a smooth map $\phi : S^1 \to M$. The Levi-Civita connection on M induces a connection on the pullback bundle ϕ^*TM, and hence a covariant operator D_ϕ. The fundamental form ω is degenerate, precisely at those ϕ for which D_ϕ has zero eigenvalue. In the first part of this paper, by using the Nash embedding theorem [4], we show that the set of points, where ω is nondegenerate, is an open set.

Loop spaces are of a particular interest to physicists working on the grand unification theory. String theory involves a theory of spinors on LM; a string structure is defined as a lifting of the structural group to an S^1-central extension of the loop group [3]. Let $G \to P \to X$ be a principal Fréchet bundle over a Fréchet manifold X that has enough smooth functions to admit a smooth partition of unity. Let $S^1 \to \tilde{G} \to G$ be an S^1-central extension of G and let $\tilde{G} \to \tilde{P} \to X$ be a lifting of the principal bundle $G \to P \to X$. Although the existence of connections on a general Fréchet bundle is in general not guaranteed, the second part of this paper verifies that every connection on the principal bundle $G \to P \to X$ together with a $\tilde{G}$-invariant connection on $S^1 \to \tilde{P} \to P$ yields a connection on $\tilde{G} \to \tilde{P} \to X$. In particular, as a corollary of this result, we prove that there exist connections on the string structures of LM.

In the last part of this paper, we give a detailed construction of Christoffel symbols on LM by using "Fourier coordinates" and compute the corresponding curvature. Both Christoffel symbols and the curvature on LM are given in terms of the Christoffel symbols on M by a Fourier type series. We hope that these constructions of Christoffel symbols and the curvature will be useful to obtain local geometrical results on LM. The proofs of our results are rather straightforward and do not use any sophisticated method of functional analysis or differential geometry.

2. Fundamental form. The tangent space $T_\phi(LM)$ at any $\phi \in LM$ can be identified with $\Gamma(\phi^*TM)$, the sections of the pullback bundle $\phi^*TM \to S^1$ of the tangent bundle $TM \to M$. For any fixed nonnegative integer r, we have an inner product $\langle\langle , \rangle\rangle_r : \Gamma(\phi^*TM)\times\Gamma(\phi^*TM) \to R$ defined by

$$\langle\langle s,s'\rangle\rangle_r = \sum_{i=0}^{r}\int_{S^1}\langle D^i s(t), D^i s'(t)\rangle dt, \tag{2.1}$$

where D is the covariant derivative of the Riemannian connection on M. This can be viewed as a Riemannian structure on LM for each $r \geq 0$. Define $\omega_\phi^{(r)} : T_\phi(LM)\times T_\phi(LM) \to R$ by $\omega_\phi^{(r)}(\alpha,\beta) = \langle\langle D\alpha,\beta\rangle\rangle_r$ and we can easily see that the fundamental form $\omega_\phi^{(r)}$ is bilinear and skew-symmetric. The energy function is defined by $e^{(r)}(\phi) = \langle\langle D\phi, D\phi\rangle\rangle_r$.

Remark 2.1. Hereafter, we simply use the notations ω and e for $\omega^{(r)}$ and $e^{(r)}$, respectively.

We verify that (i) ω is a closed form and (ii) $de + i_A\omega = 0$, where i_A is the contraction along the vector field A associated to the natural S^1-action. Since M can be isometrically embedded in R^N for some large N (by the Nash embedding theorem [4]), LM equipped with the Riemannian metric provided by $\langle\langle , \rangle\rangle_r$ is isometrically embedded in LR^N with new metric provided by $\langle\langle , \rangle\rangle_r$ for each $r \geq 0$. Therefore, it is enough to prove (i) and (ii) for $M = R^N$. In this case, the vector field A on an open subset U of LR^N is given by $(A\boldsymbol{\phi})(t) = \dot{\boldsymbol{\phi}}(t)$. Define the 1-form $\theta : U\times LR^N \to R$ as the following composition:

$$U\times LR^N \xrightarrow{A\times \mathrm{id}} LR^N\times LR^N \xrightarrow{\langle\langle , \rangle\rangle_r} R \tag{2.2}$$

so that $\theta(\phi,\alpha) = \langle\langle D\phi,\alpha\rangle\rangle_r$. Clearly θ is a smooth 1-form and $d\theta = 2\omega$. Therefore, ω is a smooth closed form. Now

$$\begin{aligned} de(\phi,\alpha) &= \lim_{s\to 0}\frac{1}{s}\left[e(\phi+s\alpha)-e(\phi)\right] \\ &= \lim_{s\to 0}\frac{1}{s}\left[\langle\langle D\phi+sD\alpha, D\phi+sD\alpha\rangle\rangle_r - \langle\langle D\phi,D\phi\rangle\rangle_r\right] \\ &= 2\langle\langle D\phi,D\alpha\rangle\rangle_r = -2\langle\langle D^2\phi,\alpha\rangle\rangle_r \\ &= -2\omega_\phi(A(\phi),\alpha) = -(i_A\omega)(\phi,\alpha). \end{aligned} \tag{2.3}$$

Hence $de + i_A\omega = 0$. The fundamental 2-form ω described above can be degenerate by the following lemma.

Lemma 2.2. *The fundamental form ω is degenerate at ϕ if and only if the corresponding covariant derivative D_ϕ has zero eigenvalue.*

Proof. The fundamental form ω is degenerate at ϕ

$$\begin{aligned} &\Longleftrightarrow \exists \alpha \in T_\phi(LM) \quad \text{such that } \omega_\phi(\alpha,\beta) = 0\ \forall \beta \in T_\phi(LM) \\ &\Longleftrightarrow \langle\langle D_\phi \alpha, \beta\rangle\rangle_r = 0 \quad \forall \beta \\ &\Longleftrightarrow D_\phi \alpha = 0. \end{aligned} \tag{2.4}$$

In other words, ω has degeneracy at ϕ if and only if there exists α in $T_\phi(LM)$ which is parallel along ϕ. For example, ω is degenerate at any closed geodesic, since $\dot{\boldsymbol{\phi}}$ is parallel along a closed geodesic ϕ. □

Theorem 2.3. *The set of points where ω is nondegenerate is an open subset of LM.*

Proof. For each $x \in M^n$ and $\tau \in LM$ that passes through x, the parallel transport $H_x(\tau)$ of $T_x(M)$ along τ is an element of the holonomy group H_x of M at x. Since the length of a vector and the angle between two vectors are preserved by the parallel transports of the Levi-Civita connection along a curve, the holonomy group at each point x of a connected orientable manifold M is a subgroup of $SO(n)$. The fundamental form ω is degenerate at some ϕ if and only if for some θ (and hence for any θ) the corresponding element of the holonomy group $H_{\phi(\theta)}$ has eigenvalue 1. Notice that every element of $SO(n)$ has eigenvalue 1 if n is odd and hence in this case ω is degenerate at every $\phi \in LM$.

So we should restrict our attention to only the even-dimensional manifolds. For every $A \in SO(n)$, let $ev_1(A)$ be the evaluation of the characteristic polynomial of A at 1. Let λ be the following composition:

$$\begin{gathered} LM \xrightarrow{i_\theta} LM \times S^1 \xrightarrow{H} SO(n) \xrightarrow{ev_1} R, \\ \phi \longmapsto (\phi,\theta) \longmapsto H_{\phi(\theta)}(\phi). \end{gathered} \tag{2.5}$$

We can easily see that λ is independent of θ and ω is degenerate at ϕ if and only if $\lambda(\phi) = 0$. Hence we have proved the theorem. □

Remark 2.4. Notice that whether the set of all nonsingular points of ω is dense or nondense in LM heavily depends on the Levi-Civita connection of M. For example, if the curvature of M is zero on a nonempty open subset of M, then the set of points, where ω is degenerate, contains a nonempty open subset of LM and hence its complement, the set of points where ω is nondegenerate, is not dense in LM. This will not be the case if the Riemannian metric on M (supposed to be a real analytic manifold) is real analytic. In this case, the set of degenerate points is a real analytic subset and hence of topological codimension 1. Then the set of nondegenerate points is dense.

3. Lifting of connections.

A Fréchet space F is called *nice* if for every open subset U of F there exists a nonzero real-valued smooth function which vanishes outside U. For example, the space of sections of a smooth vector bundle over a compact connected finite-dimensional manifold is a nice Fréchet space and hence LM is locally modeled on nice Fréchet spaces. A manifold locally modeled on nice Fréchet spaces has enough smooth functions to admit smooth partitions of unity. Hereafter, we assume that X is a manifold locally modeled on nice Fréchet spaces.

Definition 3.1. A connection on a principal G-bundle $G \longrightarrow P \xrightarrow{\pi} X$ is a smooth G-invariant splitting map $\mu : \pi^*TX \to TP$ in the exact sequence

$$0 \longrightarrow T_vP \longrightarrow TP \underset{\longleftarrow}{\overset{\mu}{\longrightarrow}} \pi^*TX \longrightarrow 0 \tag{3.1}$$

of the bundles over P where T_vP is the vertical tangent bundle.

Let $S^1 \to \widetilde{G} \to G$ be an S^1-central extension of the Fréchet Lie group G, or in other words, S^1 lies in the center of $\widetilde{G}$ and $\widetilde{G}/S^1 = G$. Let $\widetilde{G} \to \widetilde{P} \xrightarrow{\tilde{\pi}} X$ be a lifting of $G \to P \to X$. Then we have the diagram

$$\begin{tikzcd} S^1 \arrow[d] & S^1 \arrow[d] & \\ \widetilde{G} \arrow[r] \arrow[d, "\lambda"] & \widetilde{P} \arrow[d, "\lambda"] \arrow[dr, "\tilde{\pi}"] & \\ G \arrow[r] & P \arrow[r, "\pi"'] & X \end{tikzcd} \tag{3.2}$$

Of course, a connection on $G \to P \to X$ together with a connection on $S^1 \to \widetilde{P} \to P$ may not yield a connection on $\widetilde{G} \to \widetilde{P} \to X$, because a splitting map $\lambda^*TP \xrightarrow{\mu} T\widetilde{P}$ corresponding to a connection on $S^1 \to \widetilde{P} \to P$ is S^1-invariant but not necessarily $\widetilde{G}$-invariant.

Proposition 3.2. *The space $\mathscr{C}_{\widetilde{G}}(S^1 \to \widetilde{P} \to P)$ of $\widetilde{G}$-invariant connections is nonempty.*

Proof. Let $T_v^{S^1}\widetilde{P}$ be the vertical tangent bundle over $\widetilde{P}$ corresponding to the bundle $S^1 \to \widetilde{P} \to P$. We wish to show that we can produce a $\widetilde{G}$-invariant splitting map μ in

$$0 \longrightarrow T_v^{S^1}\widetilde{P} \longrightarrow T\widetilde{P} \underset{\longleftarrow}{\overset{\mu}{\longrightarrow}} \lambda^*TP \longrightarrow 0 \tag{3.3}$$

or equivalently a $\widetilde{G}$-invariant splitting map η in

$$0 \longrightarrow T_v^{S^1}\widetilde{P} \underset{\longrightarrow}{\overset{\eta}{\longleftarrow}} T\widetilde{P} \longrightarrow \lambda^*TP \longrightarrow 0. \tag{3.4}$$

Fortunately, $T_v^{S^1}\widetilde{P}$ is the trivial 1-dimensional bundle over $\widetilde{P}$. Indeed, for each $x \in \widetilde{P}$, consider the map $\{x\} \times S^1 \to \widetilde{P}$ defined by $(x, \theta) \mapsto \theta \cdot x$. Since S^1 acts locally free, the derivative of this map, defined as

$$\begin{aligned} \{x\} \times T_e(S^1) &\longrightarrow \left(T_v^{S^1}\widetilde{P}\right)_x, \\ (x, v) &\longmapsto v_x, \end{aligned} \tag{3.5}$$

provides a nonzero vector field A_v and hence $T_v^{S^1}\widetilde{P} \to \widetilde{P}$ is trivial.

Since S^1 lies in the center of $\widetilde{G}$, the action of $\widetilde{G}$ and that of S^1 commutes. Hence A_v is $\widetilde{G}$-invariant and descends to a nonzero vector field on $T_v^{S^1}\widetilde{P}/\widetilde{G} \to \widetilde{P}/\widetilde{G} = X$. Thus

$T_v^{S^1}\widetilde{P}/\widetilde{G} \to \widetilde{P}/\widetilde{G}$ is also a trivial line bundle. So it is enough to define a morphism (splitting map) γ in

$$\begin{tikzcd} X\times R = T_v^{S^1}\widetilde{P}/\widetilde{G} \arrow[rr, tail, shift right, "i"'] \arrow[dr] & & T\widetilde{P}/\widetilde{G} = E \arrow[ll, shift right, "\gamma"'] \arrow[dl] \\ & \widetilde{P}/\widetilde{G} = X & \end{tikzcd} \tag{3.6}$$

such that $\gamma\circ i = \mathrm{id}$. Since X admits partitions of unity and the convex combination of splittings is a splitting, it is enough to produce a splitting locally on

$$\begin{tikzcd} U\times R \arrow[rr, tail, shift right, "i"'] \arrow[dr] & & U\times F \cong \pi^{-1}(U) \arrow[ll, shift right, "\gamma"'] \arrow[dl] \\ & U & \end{tikzcd} \tag{3.7}$$

For each $x\in U$, let $v(x) = p_2 i(x,1) \in F$, where p_2 is the projection map. Choose a point $x_0 \in U$. Since Hahn-Banach theorem holds on any Fréchet space F, there exists a linear map $\psi : F \to R$ such that $\psi(v(x_0)) = 1$. By restricting to a smaller open subset $U_1 \subseteq U$, we can assume that $\psi(v(x)) \neq 0$ for all $x\in U$. Let $\mathcal{L}(F,R)$ be the space of linear maps from F to R. Define $\gamma : U \to \mathcal{L}(F,R)$ by

$$\gamma(x)(f) = \frac{\psi(f)}{\psi(v(x))} \quad \forall f\in F. \tag{3.8}$$

Since $\gamma(x)(v(x)) = 1$ for all x, γ yields a splitting map. This completes the proof of the lemma and immediately yields the following theorem. □

Theorem 3.3. *Every connection on the principal Fréchet bundle $G\to P\to X$ together with a $\widetilde{G}$-invariant connection on $S^1\to\widetilde{P}\to P$ yields a connection on $\widetilde{G}\to\widetilde{P}\to X$.*

Remark 3.4. The lifting of a G-connection on P to a $\widetilde{G}$-connection on $\widetilde{P}$ is not unique. The failure of uniqueness is measured by $\widetilde{G}$-invariant S^1-connections on $S^1\to\widetilde{P}\to P$ and hence by a 1-dimensional form on $\widetilde{P}$ which is $\widetilde{G}$-invariant.

Corollary 3.5. *If F is any Fréchet space with a $\widetilde{G}$-action, then every connection on $G\to P\to X$ defines at least one connection on $\widetilde{P}\times_{\widetilde{G}} F\to X$.*

Proof. The proof is the same as the finite-dimensional case. Every connection on $\widetilde{P}$ defines a connection on $\widetilde{P}\times F\to X$. Since $\widetilde{P}\times F\to\widetilde{P}\times_{\widetilde{G}}F$ is a submersion and the horizontal subspaces are mapped injectively (or in other words, horizontal subspaces of $\widetilde{P}\times F$ intersect trivially with the kernel), there is an induced connection on $\widetilde{P}\times_{\widetilde{G}}F\to X$. □

Let M^n be an even-dimensional smooth compact connected orientable manifold with the spin structure $\mathrm{Spin}(n)\to Q\to M$. Let $L\,\mathrm{Spin}(n)\to LQ\to LM$ be the associated $L\,\mathrm{Spin}(n)$-principal bundle. It is shown in [3] that such a bundle can be lifted to a new bundle $L\widetilde{\mathrm{Spin}(n)} \to \widetilde{LQ} \to LM$ provided that $p_1(M) = 0$, where $p_1(M)$ is the first Pontrjagin class of M and $L\widetilde{\mathrm{Spin}(n)}$ is the nontrivial S^1-central extension of $L\mathrm{Spin}(n)$. The spinors on LM are defined as the sections of the vector bundles $\widetilde{LQ} \times_{L\widetilde{\mathrm{Spin}(n)}} \Lambda^{\pm} \to LM$ where $L\widetilde{\mathrm{Spin}(n)}$ acts on $\Lambda^{\pm}$ by two inequivalent irreducible representations. The reader may consult [5] for the construction of such infinite-dimensional irreducible representations.

Corollary 3.6. *Any connection on $L\mathrm{Spin}(n) \to LQ \to LM$ induces connections on the spin bundles $\widetilde{LQ} \times_{L\widetilde{\mathrm{Spin}(n)}} \Lambda^{\pm} \to LM$.*

4. Christoffel symbols on LM. If M^n is an orientable Riemannian manifold and $\phi \in LM$, we can choose a frame $\{e_1, \ldots, e_n\}$ of the pullback bundle $\phi^* TM \to S^1$. Consider the Riemannian structures $\langle\langle , \rangle\rangle_{r,\phi}$ induced by the Riemannian structure of M. Let $s_1, s_2 \in T_\phi(LM)$ with $s_1 = \sum s_1^i e_i$ and $s_2 = \sum s_2^j e_j$. We can explicitly calculate $\langle\langle s_1, s_2 \rangle\rangle_{r,\phi}$. For example, if $r = 0$,

$$\begin{aligned} \langle\langle s_1, s_2 \rangle\rangle_{0,\phi} &= \int_{S^1} \langle s_1, s_2 \rangle(\theta) d\theta \\ &= \sum_{i,j=1}^{n} \int_{S^1} s_1^i(\theta) s_2^j(\theta) g_{ij}(\phi(\theta)) d\theta, \end{aligned} \tag{4.1}$$

where $g_{ij}(\phi(\theta)) = \langle e_i(\phi(\theta)), e_j(\phi(\theta)) \rangle$; and if $r = 1$,

$$\begin{aligned} \langle\langle s_1, s_2 \rangle\rangle_{1,\phi} = \sum \int_{S^1} & s_1^i(\theta) s_2^j(\theta) g_{ij}(\phi(\theta)) d\theta \\ + \sum \int_{S^1} & \dot{\boldsymbol{\phi}}(s_1^i)(\theta) \dot{\boldsymbol{\phi}}(s_2^j)(\theta) g_{ij}(\phi(\theta)) d\theta \\ + \sum \int_{S^1} & \dot{\boldsymbol{\phi}}(s_1^i)(\theta) \dot{\phi}^t(\theta) s_2^j(\theta) \Gamma_{tj}^k(\phi(\theta)) g_{ki}(\phi(\theta)) d\theta \\ + \sum \int_{S^1} & \dot{\boldsymbol{\phi}}(s_2^j)(\theta) \dot{\phi}^u(\theta) s_1^i(\theta) \Gamma_{ui}^l(\phi(\theta)) g_{lj}(\phi(\theta)) d\theta \\ + \sum \int_{S^1} & \dot{\phi}^t(\theta) \dot{\phi}^u(\theta) s_1^i(\theta) s_2^j(\theta) \Gamma_{ti}^k \Gamma_{uj}^l g_{kl}(\phi(\theta)) d\theta, \end{aligned} \tag{4.2}$$

where $\{\Gamma_{ij}^k(x)\}$ represent the Christoffel symbols of the Levi-Civita connection on M. We use the Einstein convention where the sums run through super- and subscripts. In all of our further calculations, we restrict ourselves to the case $r = 0$, and denote $\langle\langle , \rangle\rangle_{0,\phi}$ simply by $\langle\langle , \rangle\rangle_\phi$ for each $\phi \in LM$.

A connection on a Fréchet vector bundle $E \xrightarrow{\pi} X$ over a Fréchet manifold X is a rule that assigns to each point of E a complementary subspace for the vertical tangent space, called the horizontal subspace, such that the local representation of the connection is given by a smooth map Γ as follows: if π is locally $U \times G \to U$ where

$U \subseteq F$ is open and F and G are Fréchet spaces (i.e., X is locally modeled by F and G is the fiber), then the horizontal vectors consists of all $(b,c) \in F \times G$ with $c = \Gamma(u,a,b)$ where $\Gamma : U \times G \times F \to G$ is bilinear in a and b (see [2]).

By a connection on a manifold X, we always understand a connection on its tangent bundle. Unlike Banach manifolds, the existence of a connection on an arbitrary Fréchet manifold is not guaranteed. However, every connection Γ on M determines a connection on LM, also denoted by Γ, which is given locally at $\phi \in LM$ by the formula

$$\Gamma(\phi, f, g)(\theta) = \Gamma(\phi(\theta), f(\theta), g(\theta)), \tag{4.3}$$

where $f(\theta), g(\theta) \in T_{\phi(\theta)}M$ and Γ is bilinear in the last two variables.

Every $f \in LR^n$ has Fourier series

$$f \sim \sum_{p \in Z} (f_{1p}, f_{2p}, \dots, f_{np}) \alpha_p(\theta) \tag{4.4}$$

such that $\sum_p \|\vec{f}_p\|^2 p^{2k} < \infty$ for every integer k, where

$$\alpha_p(\theta) = \begin{cases} \cos p\theta & \text{if } p \le 0, \\ \sin p\theta & \text{if } p > 0. \end{cases} \tag{4.5}$$

In other words, $f \sim \sum_{p \in Z} \sum_{k=1}^n f_{kp} \alpha_p(\theta) \vec{e}_k$ where $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard basis of R^n. The numbers $\{f_{kp}\}$ are called the Fourier coordinates of $f \in LR^n$. Notice that though every element $f \in LR^n$ has Fourier coordinates which is a sequence $\{f_{kp}\}$ of real numbers, only the sequences $\{f_{kp}\}$ satisfying additional convergence conditions are coordinates for elements of LR^n (see [6, Proposition 10.2]).

By choosing a frame $\{e_1, \dots, e_n\}$ of the pullback bundle $\phi^* TM \to S^1$, just like above (beginning of Section 4), each $s \in \Gamma(\phi^* TM)$ can be assigned Fourier coordinates $\{s_{kp}\}$ where $s \sim \sum_{p \in Z} \sum_{k=1}^n s_{kp} \alpha_p(\theta) e_k$. Let Γ be a connection on LM. For a fixed $\phi \in LM$, let $\{E_{kp}(\phi)\}_{1 \le k \le n, p \in Z}$ be defined by $E_{kp}(\phi)(\theta) = \alpha_p(\theta) e_k(\phi(\theta))$. We define the Christoffel symbols $\{\tilde{\Gamma}_{kp,lq,mr}(\phi)\}$ of Γ with respect to $\{E_{kp}(\phi)\}$ by

$$\tilde{\Gamma}_{kp,lq,mr}(\phi) = \langle\langle \Gamma(\phi, E_{kp}, E_{lq}), E_{mr} \rangle\rangle_\phi, \tag{4.6}$$

where $1 \le k, l, m \le n$ and $p, q, r \in \mathbb{Z}$.

Note 1. Similar definition in the finite-dimensional case yields $\{\tilde{\Gamma}_{\beta\gamma\delta}(x)\}$ which are connected to the usual Christoffel symbols $\{\Gamma^\delta_{\beta\gamma}(x)\}$ by

$$\Gamma^\delta_{\beta\gamma}(x) = \tilde{\Gamma}_{\beta\gamma\mu}(x) (g^{-1}(x))^{\delta\mu}. \tag{4.7}$$

Now we can describe the Christoffel symbol $\tilde{\Gamma}$ in terms of the Christoffel symbols of M as follows.

Proposition 4.1. *Let $\{\Gamma^m_{kl}\}$ be the Christoffel symbols with respect to a connection Γ of M and $\{\tilde{\Gamma}_{kp,lq,mr}\}$ be the Christoffel symbols of the induced connection Γ of LM. Then*

$$\tilde{\Gamma}_{kp,lq,mr}(\phi) = \sum_{i=1}^n \int_{S^1} \alpha_p(\theta) \alpha_q(\theta) \alpha_r(\theta) \Gamma^i_{kl}(\phi(\theta)) g_{im}(\phi(\theta)) d\theta. \tag{4.8}$$

Proof. The formula for Christoffel symbols can be obtained as follows:

$$\begin{aligned}
&\langle\langle\Gamma(\phi, E_{kp}, E_{lq}), E_{mr}\rangle\rangle_\phi \\
&\quad= \int_{S^1} \langle\Gamma(\phi(\theta), E_{kp}(\phi)(\theta), E_{lq}(\phi)(\theta)), E_{mr}(\phi)(\theta)\rangle d\theta \\
&\quad= \int_{S^1} \alpha_p(\theta)\alpha_q(\theta)\alpha_r(\theta)\langle\Gamma(\phi(\theta), e_k(\phi(\theta)), e_l(\phi(\theta))), e_m(\phi(\theta))\rangle d\theta \\
&\quad= \int_{S^1} \alpha_p(\theta)\alpha_q(\theta)\alpha_r(\theta)\tilde{\Gamma}_{klm}(\phi(\theta)) d\theta \\
&\quad= \sum_{i=1}^{n} \int_{S^1} \alpha_p(\theta)\alpha_q(\theta)\alpha_r(\theta)\Gamma^i_{kl}(\phi(\theta)) g_{im}(\phi(\theta)) d\theta.
\end{aligned} \tag{4.9}$$

This completes the proof of the proposition. □

The curvature R of a connection Γ is given locally by the map

$$\begin{aligned}
R(\phi, f, g, h) = D_\phi\Gamma\{f, g, h\} - D_\phi\Gamma\{f, h, g\} \\
- \Gamma(\phi, \Gamma(\phi, f, g), h) + \Gamma(\phi, \Gamma(\phi, f, h), g)
\end{aligned} \tag{4.10}$$

which is trilinear in the last three variables. Here $D_\phi\Gamma\{f, g, h\}$ is the derivative of Γ at (ϕ, f, g) in the direction of h.

If Γ (with the curvature R) is induced by the connection Γ_M (with the curvature R_M) then

$$R(\phi, f, g, h)(\theta) = R_M(\phi(\theta), f(\theta), g(\theta), h(\theta)) \tag{4.11}$$

for all $\theta \in S^1$. As before let

$$\tilde{R}_{kp,lq,mr,ns}(\phi) = \langle\langle R(\phi, E_{kp}, E_{lq}, E_{mr}), E_{ns}\rangle\rangle_\phi. \tag{4.12}$$

Proposition 4.2. *By a similar argument as in Proposition 4.1, we have the following formula for the curvature*

$$\tilde{R}_{kp,lq,mr,ns}(\phi) = \sum_{i=1}^{n} \int_{S^1} \alpha_p(\theta)\alpha_q(\theta)\alpha_r(\theta)\alpha_s(\theta) R^i_{klm}(\phi(\theta)) g_{ir}(\phi(\theta)) d\theta, \tag{4.13}$$

where $R^i_{klm}(x)$ is defined by

$$R_M(x, e_k(x), e_l(x), e_m(x)) = R^i_{klm}(x) e_i(x) \tag{4.14}$$

for $x = \phi(\theta) \in M$.

References

[1] M. F. Atiyah, *Circular symmetry and stationary-phase approximation*, Astérisque (1985), no. 131, 43–59.

[2] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 1, 65–222.

[3] D. A. McLaughlin, *Orientation and string structures on loop space*, Pacific J. Math. **155** (1992), no. 1, 143–156.

[4] J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), 20–63.
[5] A. Pressley and G. Segal, *Loop Groups*, Oxford Mathematical Monographs, Oxford University Press, New York, 1986.
[6] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, 2nd ed., Springer-Verlag, Berlin, 2001.

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