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WEAKLY COMPACTLY GENERATED FRECHET SPACES

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ABSTRACT. It is proved that a weakly compact generated Frechet space is Lindelöf in the weak topology. As a corollary it is proved that for a finite measure space every weakly measurable function into a weakly compactly generated Frechet space is weakly equivalent to a strongly measurable function.

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1. INTRODUCTION.

If E is a weakly compactly generated Banach space then it is proved in [7] that E, with weak topology, is Lindelöf. (A topological space is said to be Lindelöf if its every open covering has a countable subcovering.) In this note we extend this result to the case when E is a weakly compactly generated

Frechet space. Also, some consequences are obtained. All locally convex spaces are taken over the field of real numbers. By a Frechet space we mean a Hausdorff, metrizable, complete locally convex space; we use the notations of [4] for locally convex spaces. E' will always denote the topological dual of a locally convex space E. A locally convex space is said to be weakly compactly generated if there exists an increasing sequence of $\sigma(E,E')$-compact subsets of E whose union is dense in E.

THEOREM 1. Let E be a weakly compactly generated Frechet space. Then $(E,\sigma(E,E'))$ is a Lindelöf space and E is a Borel subset of $(E'',\sigma(E'',E'))$, E'' being the bidual of E.

PROOF. Let $\{V_n\}$ be a sequence of 0-nbd. base having the properties:

(i) each V_n is absolutely convex and closed,

(ii) $(n+1)V_{n+1} \subset V_n$, for every n.

We take $\{A_n\}$ for an increasing sequence of weakly compact, absolutely convex subsets of E such that $\bigcup_{n=1}^{\infty} A_n = H$ is dense in E. We identify $(E,\sigma(E,E'))$ as a subspace of $R^{E'}$, with product topology. $R^{E'}$ is a subset of the compact Hausdorff space $\overline{R}^{E'}$, where $\overline{R} = [-\infty,\infty]$. For an $x \in R^{E'}$ and $y \in \overline{R}^{E'}$, $x+y \in \overline{R}^{E'}$ has the natural meaning. For a compact set $A \subset R^{E'}$ and a compact set $B \subset \overline{R}^{E'}$, $A+B$ is compact. Thus $A_k + \overline{V}_n$ is a compact subset of $\overline{R}^{E'}$ for each k and n, $\overline{V}_n$ being the closure of V_n in $\overline{R}^{E'}$. We claim that $\bigcap_{n=1}^{\infty}(H+\overline{V}_n) = E$. Since H is dense in E and V_n is a 0-nbd., $H + V_n \supset E$ for every n and so $\bigcap_{n=1}^{\infty}(H+\overline{V}_n) \supset E$. Conversely, take an $x \in \bigcap_{n=1}^{\infty}(H+\overline{V}_n)$. This means there exists a sequence $\{h_n\} \subset H$ and a sequence $\{z_n\}$ with $z_n \in \overline{V}_n$ for each n, such that $x = h_n + z_n$ for each n. Fix $n_0 \in N$ and $\epsilon > 0$. Choose an $n_1 > \max(n_0, \frac{1}{\epsilon})$ and take an $n > n_1$. Since $V_{n_0} \supset nV_n$, $|f(z_n)| \leq \frac{1}{n} < \frac{1}{n_1} < \epsilon$, for every $f \in \overset{\circ}{V}_{n_0}$ the polar of V_{n_0} ([4]). Thus $f(x-h_n) \to 0$, uniformly for $f \in \overset{\circ}{V}_{n_0}$. From this it follows that $\{h_n\}$ is Cauchy in E which is complete. If $h_n \to y$ in E it is easy to verify that, as elements of $\overline{R}^{E'}$, $x = y$. This proves the claim. Thus, in weak topology, $E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (A_k + \overline{V}_n)$ is analytic and so is Lindelöf ([6]). Also $(E'', \sigma(E'', E'))$ can be considered as a subspace of $R^{E'}$. Since $(A_k + \overline{V}_n)$ is compact in $\overline{R}^{E'}$, $(A_k + \overline{V}_n) \cap E''$ is closed in $(E'', \sigma(E'', E'))$ and so $(H + \overline{V}_n) \cap E''$ is Borel in $(E'', \sigma(E'', E'))$. Since $E = \bigcap_{n=1}^{\infty} (H + \overline{V}_n) \cap E''$, it follows that E is Borel in $(E'', \sigma(E'', E'))$.

REMARK. Similar results for Banach spaces are proved in [2, Cor. 3.2].

In the following result, some results and notations of ([3]) are used. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space, E a Hausdorff locally convex space. A function $f : X \longrightarrow E$ is called weakly measurable if $h \circ f$ is μ-measurable for every $h \in E'$. It is proved in ([2]) that if $f : X \longrightarrow E$ is weakly measurable that the image measure $\mu : \mathcal{B} \longrightarrow R$, $\nu(B) = \mu(f^{-1}(B))$, is a Baire measure on $(E, \sigma(E, E'))$, $\mathcal{B}$ being the class of all Baire subsets of $(E', \sigma(E, E'))$ ([2], [8]). Two weakly measurable functions $f_i : X \longrightarrow E$, $i = 1,2$ are said to be weakly equivalent if $h \circ f_1 = h \circ f_2$ a.e. $[\mu]$, for every $h \in E'$. If E is Frechet then $f : X \longrightarrow E$ is called strongly measurable if there exists a sequence $\{f_n\}$ of $\mathfrak{A}$-simple functions, $f_n : X \longrightarrow E$, such that $f_n \longrightarrow f$, pointwise a.e. $[\mu]$.

COROLLARY 2. Let $(X, \mathfrak{A}, \mu)$ be a finite measure space, E a weakly compactly generated Frechet space, and $f : X \longrightarrow E$ a weakly measurable function. Then f is weakly equivalent to a strongly measurable function.

PROOF. By ([3], Cor. 5) it is enough to show that image Baire measure on $(E, \sigma(E, E'))$ is tight (cf. [2]). Since $(E, \sigma(E, E'))$ is Lindelöf, Baire measures are τ-additive (normal in the terminology of [5],[8]). By ([5], Theorems 3.3, 3.4) every Frechet space is universally measurable and so every τ-smooth measure is tight. This proves the result.

REMARK. In case E is a Banach space, this result is implicit in ([2], p. 88(4), Theorem 5.4); if in addition f is bounded this is proved in ([1], p. 88).

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