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# PARAMETERS AND SOLUTIONS OF LINEAR AND NONLINEAR OSCILLATORS

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ABSTRACT. Relationship between existence of solutions for certain classes of nonlinear boundary value problems and the smallest or the largest eigenvalue of the corresponding linear problem is obtained. Behavior of the solutions, as the parameter increases, is also studied.

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## 1. INTRODUCTION.

Equations of the form

$$y''(x) + p(x)\, y(x) + \lambda q(x)\, y^n(x) = 0 \,, \tag{1.1}$$

where $\lambda$ is a parameter and n is a positive integer, arise in many physical problems, for examples, in linear (n = 1) and nonlinear (n ≠ 1) oscillation problems

[1, 2, 3, 4], and in nuclear energy distribution [5, 6]. In these problems, the parameter has physical significance, such as the energy level or the stiffness factor of the system under consideration.

In this work, relationship between existence of solutions for classes of nonlinear boundary value problems with equations of the form (1.1) and the smallest or the largest eigenvalue of the corresponding linear problem is obtained. The case of the coefficient q(x) being a negative constant has been investigated in [7]. Conditions on the coefficients of the equation, under which the solution remains bounded as the parameter increases, are obtained.

## 2. EXISTENCE OF SOLUTIONS FOR NONLINEAR BOUNDRY VALUE PROBLEMS AND EIGENVALUE OF CORRESPONDING LINEAR PROBLEMS.

In this section, relationship between existence of solutions for equations of the form (1.1) with zero boundary conditions and the smallest or largest eigenvalue of the corresponding linear problem is obtained. The analysis used here is similar to that in [7]. It would be assumed that the functions p(x) and q(x) are in the class C[0,1].

In the first two theorems, the nonlinear boundary value problem

$$y''(x) + p(x)\, y(x) - q(x)\, y^n(x) = 0\ , \tag{2.1}$$

$$y(0) = 0\ , \quad y(1) = 0\ , \tag{2.2}$$

and the corresponding linear eigenvalue problem

$$z''(x) + p(x)\, z(x) - \lambda q(x)\, z(x) = 0\ , \tag{2.3}$$

$$z(0) = 0\ , \quad z(1) = 0\ , \tag{2.4}$$

are considered.

THEOREM 2.1. If (1) $p(x) > 0$ and (2) $q(x) > 0$, then (2.1) and (2.2) has a positive solution if and only if the largest eigenvalue of (2.3) and (2.4) is positive.

PROOF. Suppose (2.1) and (2.2) has a positive solution. To show that the largest eigenvalue $\lambda_1$ of (2.3) and (2.4) is positive, let $z_1$ be a corresponding eigenfunction of (2.3) and (2.4) satisfying $z_1 \neq 0$ for $0 < x < 1$ [8]. Multiplying (2.1) by $z_1$, (2.3) by y and subtracting the equations, we get

$$y''z_1 - yz_1'' - q(x)y^n z_1 + \lambda_1 q(x)yz_1 = 0 . \tag{2.5}$$

Integration of (2.5) from 0 to 1 and the boundary conditions (2.2) and (2.4) lead to

$$-\int_0^1 q(x)y^n z_1 \, dx + \lambda_1 \int_0^1 q(x)yz_1 \, dx = 0 ,$$

therefore,

$$\lambda_1 = \frac{\int_0^1 q(x)y^n z_1 \, dx}{\int_0^1 q(x)yz_1 \, dx}$$

and $\lambda_1$ is positive.

Suppose now that the largest eigenvalue of (2.3) and (2.4) is positive. Note first that if y is positive and M denotes its maximum, then

$$y \leq M < R^{\frac{1}{n-1}} ,$$

where

$$R = \max_{x\varepsilon[0,\ 1]} \frac{p(x)}{q(x)} .$$

To apply an existence theorem for nonlinear eigenvalue problems in [9], equation (2.1) is written in the form

$$Ly = F(x, y) ,$$

where

$$Ly = -y'' + a(x)y, \quad a(x) > 0,$$

$$F(x, y) = [p(x) + a(x)]y - q(x)y^n.$$

To show that a positive solution of (2.1) and (2.2) exists, we must find curves

$u(x)$, $v(x)$ such that

$$0 < u(x) \le v(x), \quad \text{for all} \quad x \in (0, 1),$$

$$v(0) \ge 0,\ v(1) \ge 0,\ Lv \ge F(x, v),$$

$$u(0) \le 0,\ u(1) \le 0,\ Lu \le F(x, u),$$

and $a(x)$ must be chosen so that $F(x, y)$ is a monotonic increasing function of $y$ for all $(x, y)$ in the set

$$S = \{(x,y) \mid 0 \le x \le 1,\ u(x) \le y \le v(x)\}.$$

Let

$$v(x) = R^{\frac{1}{n-1}} ,$$

then

$$\begin{aligned} Lv - F(x, v) &= a(x)v - [p(x) + a(x)]v + q(x)\, v^n \\ &= v[q(x)v^{n-1} - p(x)] \\ &= R^{\frac{1}{n-1}} [q(x)R - p(x)] \\ &\ge 0 \end{aligned}$$

and $v$ satisfies all the requirements.

Let

$$u(x) = z_1(x) ,$$

where $z_1(x)$ is normalized such that

$$0 < z_1(x) \le \lambda_1^{\frac{1}{n-1}} \quad \text{and} \quad z_1(x) \le R^{\frac{1}{n-1}} , \text{ for } x \in (0, 1),$$

then

$$\begin{aligned} Lu &= -z_1'' + a(x)\, z_1 \\ &= p(x)\, z_1 - \lambda_1\, q(x)\, z_1 + a(x)\, z_1 \\ &= [p(x) + a(x)]z_1 - \lambda_1\, q(x)\, z_1 \\ &\le [p(x) + a(x)]z_1 - q(x)\, z_1^n \\ &= F(x, u). \end{aligned}$$

From the fact that

$$\frac{\partial F}{\partial y} = p(x) + a(x) - q(x)ny^{n-1} ,$$

$F(x, y)$ is increasing in $y$ in $S$ if

$$a(x) \geq q(x)ny^{n-1} - p(x) ,$$

so choose

$$a(x) \geq Q\, n\, R - p_0 ,$$

where

$$Q = \max_{x \in [0,\, 1]} q(x), \quad p_0 = \min_{x \in [0,\, 1]} p(x) .$$

By [9], the nonlinear problem (2.1) and (2.2) has at least one solution in S.

THEOREM 2.2. (1) If $p(x) > 0$, (2) $q(x) > 0$ and (3) $n$ is odd, then (2.1) and (2.2) has a negative solution if and only if the largest eigenvalue of (2.3) and (2.4) is positive.

PROOF. Suppose (2.1) and (2.2) has a negative solution. Then as in the proof of Theorem 2.1, it can be shown that if $\lambda_1$ is the largest eigenvalue of (2.3) and (2.4) and $z_1$ is a corresponding eigenfunction, then

$$\frac{\int_0^1 q(x)\, y^n\, z_1\, dx}{\int_0^1 q(x)\, y\, z_1\, dx}$$

and since n is odd, $\lambda_1$ is positive.

Conversely, suppose that the largest eigenvalue of (2.3) and (2.4) is positive. Note first that if y is negative and m denotes its minimum at say $x_0$, then

$$y'' = -p(x_0)\, m + q(x_0)m^n > 0 ,$$

$$m^n > \frac{p(x_0)}{q(x_0)} m \quad ,$$

$$m^{n-1} < \frac{p(x_0)}{q(x_0)} \quad .$$

Since (n - 1) is even,

$$-\left[\frac{p(x_0)}{q(x_0)}\right]^{\frac{1}{n-1}} < m \le y$$

and so

$$-R^{\frac{1}{n-1}} \le y \ .$$

To apply the existence theorem in [9], equation (2.1) is written in a form as in the proof of Theorem 2.1. To show that a negative solution of (2.1) and (2.2) exists, this time we must find curves $u(x)$, $v(x)$ such that

$$u(x) \le v(x) < 0, \quad \text{for} \quad x \in (0, 1),$$

$$v(0) = 0, \ v(1) = 0, \ Lv \ge F(x, v),$$

$$u(0) \le 0, \quad u(1) \le 0, \quad Lu \le F(x, u)$$

and $a(x)$ must be chosen so that $F(x, y)$ is a monotonic increasing function of $y$ for all $(x, y)$ in the set

$$S = \{(x, y) \mid 0 \le x \le 1, \quad u(x) \le y \le v(x)\} \ .$$

Let

$$u(x) = -R^{\frac{1}{n-1}} \quad ,$$

then

$$\begin{aligned} Lu - F(x, u) &= a(x)u - [p(x) + a(x)]u + q(x)\, u^n \\ &= u[q(x)\, u^{n-1} - p(x)] \\ &= -R^{\frac{1}{n-1}} [q(x)\, R - p(x)] \\ &\le 0 \end{aligned}$$

and u satisfies all the requirements.

Let

$$v(x) = z_1(x) ,$$

where $z_1(x)$ is normalized such that

$$-\lambda_1^{\frac{1}{n-1}} \le z_1(x) < 0 \quad \text{and} \quad -R^{\frac{1}{n-1}} \le z_1(x), \quad \text{for } x \in (0, 1) \quad ,$$

then

$$\lambda_1 \ge z_1^{n-1}$$

$$-\lambda_1 \, q(x) \, z_1 \ge -q(x) \, z_1^n$$

and so

$$\begin{aligned} Lv &= [p(x) + a(x)]z_1 - \lambda_1 \, q(x) \, z_1 \\ &\ge [p(x) + a(x)] \, z_1 - q(x) \, z_1^n \\ &= F(x, v) . \end{aligned}$$

From the fact that

$$\frac{\partial F}{\partial y} = p(x) + a(x) - q(x)n \, y^{n-1} ,$$

$F(x, y)$ is increasing in $y$ in $S$ if

$$\begin{aligned} a(x) &\ge q(x)ny^{n-1} - p(x) \\ &\le Q \, n \, R - p_0 , \end{aligned}$$

so let

$$a(x) \ge Q \, n \, R - p_0 .$$

It follows from [9] that the nonlinear problem (2.1) and (2.2) has at least one solution in S.

In the next Theorem, the nonlinear problem

$$y''(x) + p(x)y + q(x) \, y^n = 0 , \tag{2.6}$$

$$y(0) = y(1) = 0 , \tag{2.7}$$

and the corresponding linear eigenvalue problem

$$z''(x) + p(x)z + \lambda q(x)z = 0 , \tag{2.8}$$

$$z(0) = z(1) = 0 , \tag{2.9}$$

are considered.

THEOREM 2.3. If (1) $p(x) > 0$, (2) $q(x) > 0$ and (3) $n$ is even, then (2.6) and (2.7) has a negative solution if and only if the smallest eigenvalue of (2.8) and (2.9) is negative.

PROOF. Let

$$y(x) = -Y(x), \text{ then}$$

$$-Y''(x) - p(x)Y + q(x) [-Y(x)]^n = 0$$

and so $Y(x)$ satisfies

$$Y''(x) + p(x) Y(x) - q(x) Y^n(x) = 0 , \tag{2.10}$$

$$Y(0) = 1, \qquad Y(1) = 0 . \tag{2.11}$$

By Theorem 2.1, (2.10) and (2.11) has a positive solution $Y$ if and only if the largest eigenvalue of (2.3) and (2.4) is positive, and hence if and only if the smallest eigenvalue of (2.8) and (2.9) is negative. The conclusion of the theorem now follows.

## 3. BOUNDEDNESS OF THE SOLUTION AS THE PARAMETER INCREASES.

In this section, boundedness of the solution of

$$y''(x) + p(x)y + \lambda q(x)y^n = 0 , \tag{3.1}$$

$$y(0) = 0 , \tag{3.2}$$

as the parameter $\lambda$ increases, is studied. It would be assumed that the functions $p(x)$ and $q(x)$ are in the class $C^1[0, 1]$.

THEOREM 3.1. If (1) $p(x) > 0$, $p'(x) \le 0$, (2) $q(x) > 0$, $q'(x) \le 0$, (3) n is odd or $y \ge 0$ and (4) $\frac{y'(0)}{\sqrt{\lambda}}$ is bounded as $\lambda \to \infty$, then y is bounded as $\lambda \to \infty$.

PROOF. Multiplication of (3.1) by y' and integration of the resulting equation over [0, x] lead to

$$\frac{y'^2}{2}\Big|_0^x + p(s)\frac{y^2}{2}\Big|_0^x - \int_0^x p'(s)\frac{y^2}{2}\,ds + \lambda q(s)\frac{y^{n+1}}{n+1}\Big|_0^x - \lambda\int_0^x q'(s)\frac{y^{n+1}}{n+1}\,ds = 0,$$

$$y'^2(x) + p(x)y^2(x) - \int_0^x p'(s)y^2 ds + \frac{2}{n+1}\ \lambda q(x)y^{n+1}(x) - \frac{2}{n+1}\lambda\int_0^x q'(s)y^{n+1}\ ds = y'^2(0).$$

Therefore,

$$\frac{2}{n+1}\lambda\ q(x)\ y^{n+1}(x) \le y'^2(0),$$

$$y^{n+1}(x) \le \frac{n+1}{2}\ \frac{y'^2(0)}{\lambda q(x)}\ .$$

and the conclusion follows.

THEOREM 3.2. If (1) $p(x) > 0$, $p'(x) \ge 0$, (2) $q(x) > 0$, $q'(x) \le 0$, (3) n is odd or $y \ge 0$ and (4) $y'(0)$ is bounded as $\lambda \to \infty$, then y is bounded as $\lambda \to \infty$.

PROOF. As in Theorem 3.1, equation (3.1) is multiplied by y' and the resulting equation integrated over [0, x], obtaining

$$p(x)y^2(x) \le y'^2(0) + \int_0^x p'(s)y^2\ ds\ ,$$

$$p(x)y^2(x) \le y'^2(0) + \int_0^x p(s)y^2\ \frac{p'(s)}{p(s)}\ ds$$

and by Gronwall's inequality [10],

$$p(x)y^2(x) \le y'^2(0)\ \exp\int_0^x \frac{p'(s)}{p(s)}\ ds$$
$$= y'^2(0)\ \frac{p(x)}{p(0)}\ ,$$

therefore,

$$y^2(x) \leq \frac{y'^2(0)}{p(0)}$$

and the result follows.

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