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THE (16, 6, 2) DESIGNS

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ABSTRACT. Elementary techniques of algebraic coding theory are here used to discuss the three biplanes with k = 6. These three designs are intimately related to the (16,11) extended binary Hamming code and to one another; we systematically investigate these relationships.

We also exhibit each of the three designs as difference sets.

KEY WORDS AND PHRASES. *Biplane, Incidence Matrix, Self-orthogonal Linear Codes, (16,11) Extended Binary Hamming Code, Weight Distribution, Automorphism Groups, Difference Set.*

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§1. INTRODUCTION.

The impetus to this investigation was a still unproven conjecture of Hamada's concerning the modular rank of projective designs with classical parameters [4, 5; see also MR 48, #10842]: We wanted a non-classical set of parameters possessing several designs one of which had a doubly-transitive automorphism group. The (16,6,2) designs served and indeed an "Hamada Conjecture" is verified via elementary arguments in our Theorem 1.

We were then led easily to the relationships among the three designs and their relationship to the extended Hamming code.

Finally, in order to tie up some loose ends, we determined the automorphism groups of the two lesser known designs and were able to rather easily see that each could be represented as a difference set. One of them, in fact, in two distinct ways as a non-abelian difference set.

Specifically, our results deal with the construction of the 6-dimensional biplane with $k = 6$, $\mathcal{B}_6$, from the first-order (16,5) Reed-Muller code and its dual, the extended (16,11) Hamming code, and the subsequent constructions of the 7-dimensional biplane, $\mathcal{B}_7$, and the 8-dimensional biplane, $\mathcal{B}_8$, from the mod 2 spans of the incidence matrices of two copies of $\mathcal{B}_6$ and of $\mathcal{B}_7$, respectively.

The three non-isomorphic designs with parameters (16,6,2) have been investigated from several points of view [3, 6, 11, 12]. The vantage point of algebraic coding theory makes quite explicit certain heretofore unknown relations among them.

Hussain [6] over thirty years ago established that there were precisely three designs with parameters (16,6,2), and over twenty years ago Bruck [2] gave one of them (here designated $\mathcal{B}_6$) as a difference set in the elementary abelian group of order 16. (A referee has pointed out to us that W. Burau in 1963 independently proved that there were precisely three (16,6,2) designs; he also determined all three automorphism groups. For the details the reader should consult volume 26, pages 129 through 144, of the Hamburger Abhandlungen.) This same design arises as a rather special difference set, not only in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, but in the abelian groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$. In fact it is a difference set in twenty-four distinct ways in twelve of the groups of order 16[7]. By a theorem of Turyn's there cannot be a cyclic difference set with $v = 16$; it has not been generally recognized, however, that there are two distinct difference sets in $\mathbb{Z}_2 \times \mathbb{Z}_8$ (both given by Turyn). One, namely,

$$\{(0,0),(0,1),(0,2),(0,5),(1,0),(1,6)\},$$

yields $\mathcal{B}_6$ again.

The other difference set in $\mathbb{Z}_2 \times \mathbb{Z}_8$ given by Turyn, namely,

$$\{(0,0),(0,1),(1,0),(1,2),(1,5),(1,6)\},$$

yields $\mathcal{B}_7$. We have checked by a combination of hand and computer calculation that $\mathcal{B}_7$'s automorphism group has only $\mathbb{Z}_2 \times \mathbb{Z}_8$ acting regularly on its points; thus it arises as a difference set in no other group of order 16. The automorphism group of $\mathcal{B}_7$ has order $48 \cdot 16$ and the subgroup of order 48 fixing a block acts as the full group of symmetries of the cube when the six points of the block are properly identified with the six faces of the cube. This subgroup is thus a central extension of Sym(4) by a group of order 2.

$\mathcal{B}_8$ does not arise as an abelian difference set but it does arise in two (and precisely two) distinct ways as a non-abelian difference set. The details are as follows:

Let $G = Q \times \mathbb{Z}_2$ where Q is the quaternion group of order 8. Writing multiplicatively with $\mathbb{Z}_2 \simeq \langle\phi\rangle$, using the usual notation for Q, and agreeing that $q\phi$ means (q,ϕ) whereas q means (q,e), $\{1,i,j,k,\phi,-\phi\}$ is easily seen to be a difference set in $Q \times \langle\phi\rangle$. One can, with a slightly more tedious computation, check that the rank of the incidence matrix of the arising (16,6,2) design is 8 (over the two-element field).

With the same notation as above but now assuming that ϕ is of order 4 generating the center of a group of order 16 containing Q, where $\phi^2 = -1 \in Q$, we get still another difference set which is formally identical. Hence the arising design is again $\mathcal{B}_8$.

Again by a combination of hand and computer calculation we have checked that $\mathcal{B}_8$'s automorphism group has only the above two groups acting regularly on its points. This automorphism group has order $24 \cdot 16$ and the subgroup of order 24 fixing a block has one six point orbit (not a block but the complement of the union of the two blocks which it fixes). With these six points properly identified with the six faces of the cube, the group acts as the inverse image of Alt(4) in the central extension of Sym(4) by a cyclic group of order 2 described above.

The clutch of difference sets referred to and displayed above (a dozen in number) leave but two non-abelian groups of order 16 without difference sets. Robert E. Kibler, [7], has enumerated all (16,6,2) difference sets. Of the fourteen groups of order 16, all but the cyclic and dihedral groups possess difference sets, and there are twenty-seven in all. Those that we have not referred to all involve the design $\mathcal{B}_6$.

§2. DEFINITIONS.

A t-(v,k,λ) design on a v-set S (a finite set of cardinality v whose elements are called points) is a collection, D, of k-subsets of S (called blocks) such that every t-subset of S is contained in precisely λ elements of D. A projective design is a 2-(v,k,λ) design in which the number of blocks and points is the same. A biplane is a projective design with λ = 2. We have therefore that each point of a biplane lies in k blocks, that every pair of blocks meets in two points, and that $k(k-1) = 2(v-1)$.

Throughout F will denote the field with two elements. A binary linear (n,k) code A is a k-dimensional subspace of F^n. For $a \in A$, $wgt(a) = |\{i|a_i = 1\}|$ is the weight of the code vector a, where $a = (a_1,\dots,a_n)$ and $|X|$ represents the cardinality of the set X. $A^{\perp} = \{b \in F^n | a \cdot b = 0 \text{ for all } a \in A\}$ is the orthogonal of A, where $a \cdot b$ is the usual dot product of two vectors. The orthogonal of an (n,k) linear code is an (n,n-k) linear code. A is self-orthogonal if $A \subseteq A^{\perp}$ and self-dual if $A = A^{\perp}$. The minimum weight of A is $d(A) = \text{Min}\{wgt(a)|a \in A, a \neq 0\}$. For $a \in A$, $\text{Supp}(a) = \{j|a_j = 1\}$ is the support of the vector a; since we will only consider binary codes the vector and its support carry precisely the same information and we will occasionally not distinguish the vector and its support.

If A is a binary self-orthogonal code, then the function $a \to \frac{1}{2} wgt(a)$ (mod 2) is not only well-defined but a linear transformation from A to F. Its kernel consists of those vectors in A whose weight is congruent to zero modulo four. We denote this "subcode" of A by Ker A . Clearly

Ker A = A or Ker A is of codimension 1 in A according to whether or not the functional, $a \to \frac{1}{2}\,\mathrm{wgt}(a)$ is zero or not.

Throughout H will denote the extended binary (16,11) Hamming code. Its "weight distribution" is

$$x^0 + 140x^4 + 448x^6 + 870x^8 + 448x^{10} + 140x^{12} + x^{16},$$

where nx^i denotes the presence of n code vectors of weight i. $H^\perp$ is the (16,5) first order Reed-Muller code. It is self-orthogonal with weight distribution

$$x^0 + 30x^8 + x^{16}.$$

The automorphism group of H and $H^\perp$ is the subgroup of Sym(16) (acting on the sixteen coordinate places) which leaves the subspace H (and hence $H^\perp$) fixed. It is the triply-transitive affine group of F^4. It follows that the thirty weight-8 vectors of $H^\perp$ form a 3-(16,8,3) design, a result that one could also obtain from the weight distributions and the Assmus-Mattson Theorem.

The incidence matrix of a projective design is the v x v matrix $(a_{B,p})$, where $a_{B,p}$ is 1 if the point P and block B are incident and 0 otherwise. The p-rank (i.e., its rank over the field with p elements) of the incidence matrix of a projective design is a function of the parameters of the design, unless $p|(k-\lambda)$, in which case it may depend on the block structure of the design [4]. Hence, the only prime of interest in discussing biplanes with k = 6 is p = 2. This fact is what forced our restriction to binary codes.

For an incidence matrix, M, of such a biplane, Sp(M) will denote its row space over F. Thus Sp(M) (or Sp($\mathcal{B}$) if we are naming the biplane rather than its incidence matrix) will be a binary $(16,\ell)$ code where ℓ will depend on $\mathcal{B}$. As we shall see, it carries a great deal of information concerning the biplane.

There are, up to isomorphism, precisely three (16,6,2) designs, [6]. They will each appear in what follows and will be denoted by $\mathcal{B}_6$, $\mathcal{B}_7$, and $\mathcal{B}_8$, the subscript denoting the dimension of $Sp(\mathcal{B})$ over F. A quite remarkable fact about these three biplanes is that tokens can be so chosen that

$$Sp(\mathcal{B}_6) \subset Sp(\mathcal{B}_7) \subset Sp(\mathcal{B}_8)$$

We know of no other instance in which such a "nesting" of designs occurs.

As we shall see in §§'s 4 and 5, $Sp(\mathcal{B}_7)$ contains 8 copies of $\mathcal{B}_7$ and 8 copies of $\mathcal{B}_6$ while $Sp(\mathcal{B}_8)$ contains 192 copies of $\mathcal{B}_8$, 288 copies of $\mathcal{B}_7$, and 96 copies of $\mathcal{B}_6$.

§3. <u>THE MOD 2 SPAN</u>.

PROPOSITION 1: Let M be the incidence matrix of a projective (16,6,2)-design and R be the row-space over F of M. Then $R \subseteq R^{\perp}$, i.e. R is self-orthogonal, and $d(R^{\perp}) \geq 4$.

PROOF: The fact that R is self-orthogonal is immediate from the design parameters. Suppose $v \in R^{\perp}$ and $p \in Supp(v)$. Then each of the six blocks through p meets Supp(v) evenly and hence at least once more. It follows that

$$2(|Supp\ v|-1) \geq 6$$

and hence that $|Supp\ v| \geq 4$; i.e., $d(R^{\perp}) \geq 4$.

REMARK: Since the all-one vector is in $R^{\perp}$, no vector in $R^{\perp}$ can have weight 14. As we shall soon see R contains the all-one vector also, and hence the weights in $R^{\perp}$ must all be even. Moreover, the vectors of weight-4 in $R^{\perp}$ are the "ovals" of the design in the sense of [1].

LEMMA 1: Let S be a set of three points of a projective (16,6,2)-design and suppose B_1, B_2 and B_3 are three blocks meeting S in its three 2-subsets. Then the mod 2 sum of B_1, B_2 and B_3 is a vector of weight 6 or 10. Moreover, if S is contained in a block C and the weight is 10, the weight-10 vector is the complement of C.

PROOF: Let x_i, $0 \leq i \leq 3$ be the number of points not in S through which there pass i of the three blocks. Then

$$x_0 + x_1 + x_2 + x_3 = 13$$
$$x_1 + 2x_2 + 3x_3 = 3 \cdot 4 = 12$$
$$x_2 + 3x_3 = 3 \cdot 1 = 3$$

Since x_3 can be at most 1, the two solutions are

$$x_0 = 3,\ x_1 = 9,\ x_2 = 0,\ x_3 = 1$$

and

$$x_0 = 4,\ x_1 = 6,\ x_2 = 3,\ x_3 = 0.$$

Thus the weight of the mod 2 sum is either 6 or 10. Moreover, if $S \subseteq C$ and the weight is 10, the three points through which none of the blocks pass must be C-S and the weight-10 vector must consist of the ten points not in C.

LEMMA 2: Let M be the incidence matrix of a projective (16,6,2)-design and R the row space over F of M. Then the all-one vector is in R.

PROOF: Suppose not. Let C be any block. Then for each 3-subset, S, of C the three blocks meeting S twice sum to a vector of weight 6, $v(S)$ where the support of $v(S)$ is disjoint from C. For such a

3-subset S let S' be the 3-subset of points not in C which the three blocks meeting S twice cover exactly twice. (See the proof of Lemma 1.) Now no block covers S' for then it could not be orthogonal to C and $v(S)$ unless the sum of it and C were a weight-8 vector disjoint from supp($v(S)$), an impossibility since this would produce a weight-14 vector in R. Thus there are three other blocks B_1, B_2 and B_3 meeting S' twice. Now suppose $B_i \cap S$ were non-empty. Then each of the three blocks meeting S twice would meet B_i twice in $S \cup S'$. Hence $B_i \cap \text{Supp}(v(S)) = \emptyset$ and we would have a weight-8 vector, $C + B_i$ disjoint from Supp($v(S)$), again a contradiction. Thus B_1, B_2 and B_3 are the three vectors meeting $C - S$ twice and their mod 2 sum is necessarily $v(S)$; i.e., $v(S) = v(C-S)$. Now for any 3-subset S_1 of C other than S or $C - S$, $v(S_1) \neq v(S)$ for otherwise one could find three blocks through two points of S'. It follows that we have $10 = \frac{1}{2}\binom{6}{3}$ weight-6 vectors whose supports are on the complement of C. Moreover, since they are an orthogonal set of vectors and no two of the supports can meet twice (since as above we would produce a vector of weight 14), every two supports meet four times; i.e. we have a (10,6,4)-design on the complement of C. But these are impossible parameters for a projective design.

LEMMA 3: Let M be the incidence matrix of a projective (16,6,2)-design and R the row space over F of M. Then R contains at least thirty weight-8 vectors.

PROOF: If C is a block then the mod 2 sum of C and any other block is a weight-8 vector. This yields fifteen weight-8 vectors. Since the all-one vector is in R one gets fifteen more by complementation.

Thus there are at least thirty. (They are clearly distinct since the intersections of their supports with C are.)

THEOREM 1: Let M be the incidence matrix of a projective (16,6,2)-design and R the row space over F of M. Then $6 \leq \dim R \leq 8$ and, moreover there is a unique such design with $\dim R = 6$. We denote this biplane by $\mathcal{B}_6$.

PROOF: That $\dim R \leq 8$ follows immediately from the fact (Proposition 1) that $R \subseteq R^{\perp}$. That $6 \leq \dim R$ is a consequence of Lemmas 2 and 3 since R has at least 16 weight-6 vectors, 30 weight-8 vectors and 16 weight-10 vectors - and, of course, the zero vector and all-one vector. Now if $\dim R = 6$, the weight distribution is determined and Ker R consists of the zero vector, all-one vector, and the thirty weight-8 vectors. There is a unique such code, the first order Reed-Muller code. It is the orthogonal to the extended (16,11) Hamming code, H. Take any weight-6 vector of H, v say. Then $Fv + H^{\perp}$ is a (16,6) code. Since no vector of weight eight in $H^{\perp}$ is disjoint from v (there are no weight-14 vectors in H) each weight-8 vector meets Supp(v) in two or four points. Clearly, 15 meet in 2 and 15 in 4. Thus $Fv + H^{\perp}$ has 16 weight-6 vectors and 16 weight-10 vectors. We have before us all the vectors in $Fv + H^{\perp}$. Thus two distinct weight-6 vectors must have their supports meeting exactly twice because of the lack of weight-4 and weight-12 vectors. It follows that we have a (16,6,2) design and that any such is obtained in this manner from the (16,11) extended Hamming code. Since the extended Hamming code's automorphism group is transitive on the weight-6 vectors, the (16,6,2)-design of dimension six is unique.

REMARK 1. There also are unique (16,6,2) designs of dimensions 7 and 8. They will appear shortly as $\mathcal{B}_7$ and $\mathcal{B}_8$.

REMARK 2. The automorphism group of this (16,6,2)-design is doubly-transitive, a fact one can establish from its description above. A purely combinatorial description that also establishes this fact was communicated to us by Richard M. Wilson. We sketch it here: Let the sixteen points of the design be the entry positions of a 4 by 4 matrix. The sixteen blocks are determined by these entry positions being the entry positions in its row and column other than itself. (This description of the design appears in Crelle's Journal, 70(1869), page 182; Jordan there obtains the group of the design by actually writing down generating permutations.) See the figure:

		x	
x	x	.	x
		x	
		x	

This clearly yields a (16,6,2) design with a transitive automorphism group. On the other hand consider the point set given by the fifteen edges of K_6 (the complete graph on six vertices) plus a point labelled ∞. The blocks are described as follows: Those not containing ∞ consist of the six edges of two disjoint triangles of K_6 of which there are clearly $\frac{1}{2}\binom{6}{3} = 10$. Those containing ∞ consist of ∞ and the five edges emanating from a fixed vertex of K_6. One verifies easily that this yields a (16,6,2)-design and the subgroup of its automorphism group leaving ∞ fixed is Sym(6) acting naturally on K_6 and hence transitively on the other fifteen points. It remains to show that these two designs are isomorphic. The isomorphism is explicitly given by the following table:

∞	1,2	1,3	2,3
4,5	3,6	2,6	1,6
4,6	3,5	2,5	1,5
5,6	3,4	2,4	1,4

where we have labelled the vertices of K_6 with 1 through 6 and an entry {a,b} means the edge {a,b} corresponds to the entry position in which it appears.

COROLLARY 1: The 448 weight-6 vectors of the extended (16,11) binary Hamming code split naturally into 28 disjoint isomorphic (16,6,2) designs. Thus we can explicitly construct 2-(16,6,λ) designs for $\lambda = 2i$, $i = 1,2,\ldots,28$.

COROLLARY 2: The weight distribution of $(Sp\mathcal{B}_6)^{\perp}$ is

$$x^0 + 60x^4 + 256x^6 + 390x^8 + 256x^{10} + 60x^{12} + x^{16}$$

and hence $\mathcal{B}_6$ has 60 ovals.

PROOF: Knowing the weight distribution of $Sp\mathcal{B}_6$ allows one to easily calculate that of its dual via the MacWilliams equations. The 60 weight-4 vectors are the ovals of the biplane, cf. [1].

§4. $\mathcal{B}_7$.

Setting $A_6 = Sp\mathcal{B}_6$ we can arrange things as we've just seen so that

$$H^{\perp} = \text{Ker } A_6 \subset A_6 \subset A_6^{\perp} \subset H$$

and we have the weight distribution of each of these four subspaces of F^{16}. In particular there are 240 weight-6 vectors in $A_6^{\perp}$ which are not among the 16 weight-6 vectors of A_6.

LEMMA 4: Let v be a weight-6 vector of $A_6^{\perp}$ not in A_6. The weight distribution of $A_6 + Fv = A_7$ is

$$x^0 + 4x^4 + 32x^6 + 54x^8 + 32x^{10} + 4x^{12} + x^{16}.$$

PROOF: A_7 is clearly self-orthogonal and of dimension 7. Since Supp v cannot intersect each of the 16 blocks of $\mathcal{B}_6$ exactly twice, there is a $w \in A_6$ of weight 6 with $|\text{Supp } w \cap \text{Supp } v| = 0$ or 4. Let b be the weight-4 vector which is either $v+w$ or its complement.

The choice of any three 1's of b will determine three weight-8 vectors of $H^{\perp}$, since the 30 weight-8 vectors form a 3-(16,8,3) design. These three vectors must have precisely four 1's in common and, moreover, they must be the four 1's of b, in order for b to be orthogonal to each of the three vectors. Hence there exist four weight-4 vectors whose sum is the all-one vector.

Now Ker A_7 is 6-dimensional and is clearly $H^{\perp} + Fb$. Thus the four weight-4 vectors we found are all there are, and the weight distribution of Ker A_7 is

$$x^0 + 4x^4 + 54x^8 + 4x^{12} + x^{16}.$$

Since A_7 contains at least the 32 weight-6 vectors (the 16 from A_6 and the 16 from $H^{\perp} + Fv$) and hence at least 32 weight-10 vectors the result is established.

Retaining the notation above, let Q be the 4-dimensional subspace generated by the four weight-4 vectors of A_7; the weight distribution of Q is clearly

$$x^0 + 4x^4 + 6x^8 + 4x^{12} + x^{16}.$$

Now suppose sixteen of A_7's weight-6 vectors form a biplane $\mathcal{B}$ isomorphic to $\mathcal{B}_6$. Then $\text{Ker Sp}\mathcal{B} \subseteq \text{Ker } A_7$ and $\dim(\text{Ker Sp}\mathcal{B} \cap Q) = 5 + 4 - 6 = 3$. Thus $P = \text{Ker Sp}\mathcal{B} \cap Q$ has weight distribution

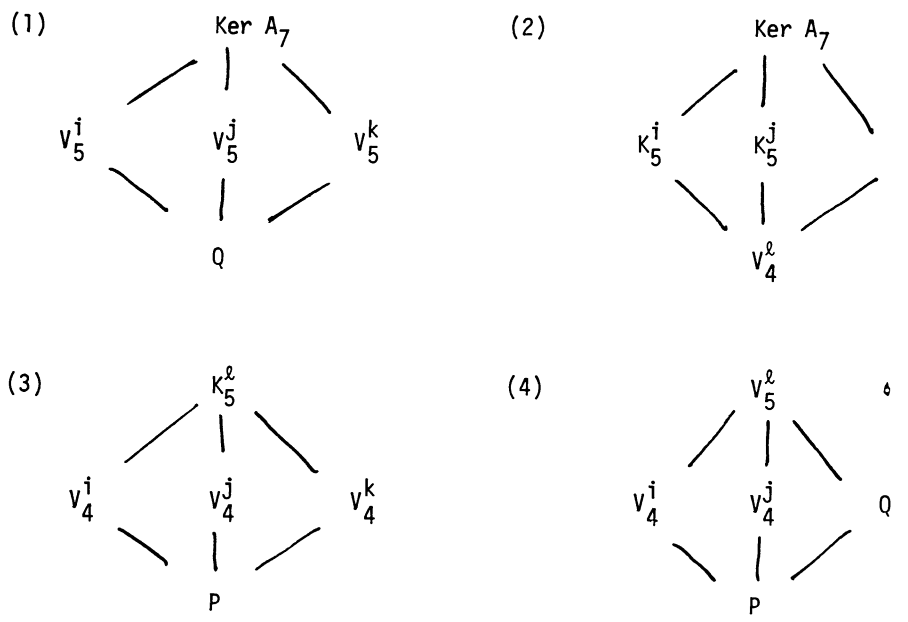
$$x^0 + 6x^8 + x^{16}.$$

Now Ker A_7 is 6-dimensional and P is 3-dimensional. Thus there are seven 5-dimensional and seven 4-dimensional subspaces between P and Ker A_7. Q is one of the 4-dimensional subspaces and it is contained in precisely three of the 5-dimensional subspaces. It follows that the remaining four 5-dimensional subspaces contain no weight-4 vectors and hence must be equivalent to $H^\perp$. Each of these four subspaces will split the 32 weight-6 vectors of A_7 into two biplanes isomorphic to $\mathcal{B}_6$. We record this as

PROPOSITION 2: The 32 weight-6 vectors of A_7 can be split into the disjoint union of two isomorphic copies of $\mathcal{B}_6$ in exactly four distinct ways. There are, thus, precisely eight copies of $\mathcal{B}_6$ to be found among these 32 vectors.

We next analyze these eight $\mathcal{B}_6$'s in order to construct $\mathcal{B}_7$.

Again with the notation as above let V_4^i, $1 \leq i \leq 6$, denote the six 4-dimensional subspaces other than Q, and V_5^i, $1 \leq i \leq 3$, the three 5-dimensional subspaces other than the four equivalent to $H^\perp$; we denote these four by K_5^i where $1 \leq i \leq 4$. Since $V_5^i \supset Q$, it has weight distribution $x^0 + 4x^4 + 22x^8 + 4x^{12} + x^{16}$, and since V_4^i has no weight-4 vectors, it has weight distribution $x^0 + 14x^8 + x^{16}$. The following diagrams illustrate the relationships between these subspaces:



Now, (1), (3), and (4) are clearly true. To establish (2), it suffices to note that $V_4^\ell + F\,b = V_5^k$, where wgt(b) = 4, and then (1) implies the other two subspaces must be K_5^i's. Thus $K_5^i \cap K_5^j \cap K_5^k = P$, where $1 \leq i < j < k \leq 4$.

Adjoining a common weight-6 vector, v, from A_7 to each of the K_5^i's, $1 \leq i \leq 4$, will yield four isomorphic copies of $\mathcal{B}_6$, say $\mathcal{B}_6^{i+}$, $1 \leq i \leq 4$, having a common intersection of four weight-6 vectors, since their spans intersect in a subspace of dimension 4, $P + F\,v$, having weight distribution $x^0 + 4x^6 + 6x^8 + 4x^{10} + x^{16}$. In this way, P partitions the 32 weight-6 vectors of A_7 into eight <u>packets</u> of four vectors each. Let the four

$\mathcal{B}_6$'s complementary to the four $\mathcal{B}_6^{i+}$'s be denoted by $\mathcal{B}_6^{i-}$, $1 \le i \le 4$, where two $\mathcal{B}_6$'s are said to be complementary if they have no blocks in common.

Any two noncomplementary $\mathcal{B}_6$'s have eight blocks in common (the elements of two packets), since the kernels of their mod-2 spans intersect in a V_4^i and the weight distribution of $V_4^i + F\,v$ is necessarily $x^0 + 8x^6 + 14x^8 + 8x^{10} + x^{16}$, where v is a weight-6 vector corresponding to a common block of the noncomplementary pair of $\mathcal{B}_6$'s. Inasmuch as any three K_5^i's intersect in P, any three pairwise noncomplementary $\mathcal{B}_6$'s have one packet in common and their spans intersect in a subspace with weight distribution $x^0 + 4x^6 + 6x^8 + 4x^{10} + x^{16}$.

The following diagram and equations summarize these relationships. A "+" or "-" in the diagram symbolizes the presence of the packet represented by column j in $\mathcal{B}_6^{i+}$ or $\mathcal{B}_6^{i-}$, respectively, where i represents the row. In the equations, $x_i \in \{+,-\}$, and the number of blocks in an intersection of $\mathcal{B}_6$'s is given.

+	+	+	+	-	-	-	-
+	+	-	-	+	+	-	-
+	-	+	-	+	-	+	-
+	-	-	+	-	+	+	-

$$|\mathcal{B}_6^{ix_i} \cap \mathcal{B}_6^{jx_j}| = 8, \text{ where } 1 \le i < j \le 4$$

$$|\mathcal{B}_6^{ix_i} \cap \mathcal{B}_6^{jx_j} \cap \mathcal{B}_6^{kx_k}| = 4, \text{ where } 1 \le i < j < k \le 4$$

$$\left|\bigcap_{i=1}^{4} \mathcal{B}_6^{ix_i}\right| = \{{4 \atop 0}\} \iff \sum_{i=1}^{4} x_i 1 \equiv \{{0 \atop 2}\} \bmod 4$$

PROPOSITION 3: $A_7 = Sp(\mathcal{B}_7)$. Moreover, the 32 weight-6 vectors of A_7 can be split into the disjoint union of two isomorphic copies of $\mathcal{B}_7$ in exactly four ways, thereby yielding the eight $\mathcal{B}_7$'s which can be found in the collection of 32 vectors.

PROOF: In the above diagram, two packets (columns) will be called complementary if there does not exist a $\mathcal{B}_6$ containing both packets. Using the notation given above, the complementary packets are precisely those whose column indices sum to nine. Any two noncomplementary packets are contained in precisely two $\mathcal{B}_6$'s. Thus if one permutes the columns of any two complementary packets in the diagram one obtains a new diagram which indicates the packet structure of eight new biplanes. Any odd permutation of complementary packets will result in the same new packet structure, while an even permutation produces the original packet structure. Since there exist precisely eight $\mathcal{B}_6$'s in A_7, of dimension 7, the eight new biplanes must all be $\mathcal{B}_7$'s.

One can show that every $\mathcal{B}_7$ in A_7 is composed of the packets determined by P and every $\mathcal{B}_7$ intersects four $\mathcal{B}_6$'s in three packets and the other four in one packet [9]. Thus two packets occur together in a $\mathcal{B}_7$ if and only if they occur together in a $\mathcal{B}_6$. Hence if one replaces a packet of a $\mathcal{B}_6$ with a noncomplementary packet, one obtains a collection of four packets, which includes two complementary packets, and so cannot comprise a $\mathcal{B}_7$. Therefore, A_7 contains precisely eight isomorphic copies of $\mathcal{B}_7$.

COROLLARY: $\mathrm{Aut}(A_7)$ has order $2^{11}\cdot 3$.

PROOF: Since $|\mathrm{Aut}(\mathcal{B}_7)| = 48\cdot 16$ and there are eight $\mathcal{B}_7$'s in A_7

the corollary follows from the fact that $Sp(\mathcal{B}_7) = A_7$ for any $\mathcal{B}_7$ in A_7.

§5. $\underline{\mathcal{B}_8}$.

We are now in a position to arrange the two biplanes, $\mathcal{B}_6$ and $\mathcal{B}_7$, so that $Sp(\mathcal{B}_6) = A_6 \subseteq A_7 = Sp(\mathcal{B}_7)$ and we have the following nesting

$$H^{\perp} \subset A_6 \subset A_7 \subset A_7^{\perp} \subset A_6^{\perp} \subset H.$$

We know the weight distribution of all these subspaces. The only one we have not yet recorded is that of $A_7^{\perp}$. Again, it is easily calculated via the MacWilliams equations and is

$$x^0 + 28x^4 + 128x^6 + 198x^8 + 128x^{10} + 28x^{12} + x^{16}.$$

There are, thus, 28 ovals in a $\mathcal{B}_7$.

There are 96 weight-6 vectors in $A_7^{\perp}$ that are not in A_7. If v is any one of these $A_7 + Fv$ is a self-dual (16,8) code containing weight-6 vectors but no weight-2 vectors, this last assertion since $A_7 + Fv = A_8 \subseteq H$. There is a unique such code [8] and its weight distribution is

$$x^0 + 12x^4 + 64x^6 + 102x^8 + 64x^{10} + 12x^{12} + x^{16}.$$

Since $Sp(\mathcal{B}_8)$ contains weight-6 vectors but no weight-2 vectors by Prop. 1, $A_8 = Sp(\mathcal{B}_8)$. The nesting we now have is

$$H^{\perp} \subset A_6 \subset A_7 \subset A_8 = A_8^{\perp} \subset A_7^{\perp} \subset A_6^{\perp} \subset H.$$

Now the 64 weight-6 vectors of A_8 can be split as the disjoint union of four $\mathcal{B}_6$'s or (as we have seen in §4) as the disjoint union of four $\mathcal{B}_7$'s.

A computer calculation has shown that they can be split as a disjoint union of four $\mathcal{B}_8$'s. Since $A_8 = A_8^{\perp}$, $\mathcal{B}_8$ has 12 ovals.

We next calculate the number of distinct copies of $\mathcal{B}_6$ and $\mathcal{B}_7$ to be found among the 64 weight-6 vectors of A_8.

PROPOSITION 4: Ker A_8 contains precisely 24 subspaces equivalent to Ker A_6 and 18 subspaces equivalent to Ker A_7.

PROOF: Suppose a Ker A_6 is contained in Ker A_8. Let $b \in$ Ker A_8 be of weight 4. Since Ker A_6 + Fb is a Ker A_7, each Ker A_6 is contained in three Ker A_7's contained in Ker A_8, and the twelve weight-4 vectors of Ker A_8 are split into three classes of four disjoint weight-4 vectors and form a 1-(16,4,3) design. A weight-4 vector from a class will meet precisely two weight-4 vectors from another class exactly twice and thereby determine a 1-(8,4,3) design on the eight points contained in the supports of the two weight-4 vectors of the second class. These six vectors, as well as the remaining six, can be split into three collections, each containing two disjoint weight-4 vectors. Hence the twelve weight-4 vectors of Ker A_8 can be decomposed into precisely nine distinct collections, each containing four disjoint weight-4 vectors.

Let Q, as previously, be the subspace spanned by a collection of four disjoint weight-4 vectors. We shall count $\{(Q,\text{Ker } A_7) \mid Q \subset \text{Ker } A_7 \subset \text{Ker } A_8\}$. Since each Ker A_7 determines a unique Q, 9q = x, where q is the number of Ker A_7's containing a given Q and x is the number of Ker A_7's contained in Ker A_8. Adjoining a new weight-4 vector to a Q establishes a subspace of dimension 5 with eight weight-4 vectors. Thus Q is contained in two such 5-dimensional subspaces, each in turn being contained in two distinct 6-dimensional subspaces and a third which contains both. Hence q = 2 and so x = 18.

By counting $\{(\text{Ker } A_6, \text{Ker } A_7) | \text{Ker } A_6 \subset \text{Ker } A_7 \subset \text{Ker } A_8\}$, one easily sees that 24 subspaces equivalent to a Ker A_6 are contained in Ker A_8, and so there exist $4 \cdot 24 = 96$ $\mathcal{B}_6$'s in Sp($\mathcal{B}_8$).

Since a Ker A_7 determines exactly $8 \cdot 2 = 16$ distinct $\mathcal{B}_7$'s in a Sp($\mathcal{B}_8$) and each $\mathcal{B}_7$ is determined by a unique Ker A_7, there exist $18 \cdot 16 = 288$ $\mathcal{B}_7$'s in Sp($\mathcal{B}_8$).

We note that $288 = 3 \cdot 96$ since each Ker A_7 also determines 16 $\mathcal{B}_6$'s but a given $\mathcal{B}_6$ is determined by the three Ker A_7's containing Ker A_6.

REMARKS: 1. Such a nesting of designs is impossible for projective planes of the same order, since in the rowspace of the incidence matrix over F_p, where p divides the order of the plane, the supports of the minimum weight code vectors are precisely the blocks (lines) of the plane [10]. Of the three biplanes discussed here, only $\mathcal{B}_6$ enjoys this property.

2. Since Aut A_8 is known [8] and of order $2^{13} \cdot 3^2$ and Aut $\mathcal{B}_8$ is of order $2^7 \cdot 3$ there are 192 distinct $\mathcal{B}_8$'s contained in A_8.

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