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# CONVEXITY, BOUNDEDNESS, AND ALMOST PERIODICITY FOR DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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ABSTRACT. There are three kinds of results. First we extend and sharpen a convexity inequality of Agmon and Nirenberg for certain differential inequalities in Hilbert space. Next we characterize the bounded solutions of a differential equation in Hilbert space involving and arbitrary unbounded normal operator. Finally, we give a general sufficient condition for a bounded solution of a differential equation in Hilbert space to be almost periodic.

KEY WORDS AND PHRASES. *Differential equations in Hilbert space, Convexity inequality, Self-adjoint operators, Bounded solutions, Almost periodic solutions.*

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1. INTRODUCTION. Let $S_1$, $S_2$ be two commuting self-adjoint operators on a complex Hilbert space $H$. Let $u : [a,b] \to H$ satisfy the inequality

$$\|du(t)/dt - (S_1 + iS_2)\, u(t)\| \le \phi(t)\, \|u(t)\| , \quad a \le t \le b , \tag{1.1}$$

where $\int_a^b \phi(t)dt \le c < 1/2$. We shall show that this implies the convexity inequality

$$\|u(t)\| \le K_c \|u(a)\|^{\frac{b-t}{b-a}} \|u(b)\|^{\frac{t-a}{b-a}} ,$$

which holds for some constant $K_c$ and all $t \in [a,b]$. S. Agmon and L. Nirenberg [1] first proved this assuming $c = 2^{-3/2}$ ; recently S. Zaidman [7] extended it to weak solutions of (1.1). Our results apply to weak solutions and to the range of values $0 < c < 1/2$ ; moreover, we obtain a smaller constant $K_c$ than did these previous authors. This result is presented in Section 2.

Section 3 is devoted to obtaining the structure of the set of all bounded solutions of

$$du(t)/dt = (S_1 + iS_2)u(t) \qquad (-\infty < t < \infty).$$

The results generalize and improve a recent result of Zaidman [8].

In Section 4 we study almost periodic solutions of the inhomogeneous equation

$$du(t)/dt = Au(t) + f(t) \qquad (-\infty < t < \infty) ;$$

here $A$ is a closed linear operator on $H$ and $f$ is an $H$-valued function. Under a finite dimensionality assumption we show that bounded solutions are almost periodic. This generalizes the results obtained by Zaidman in [6].

2. A CONVEXITY THEOREM. Let $u$ map the real interval $[a,b]$ into a complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. Let $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ be a closed, densely defined linear operator. $u$ is a *strong solution* of

$$\|du(t)/dt - Bu(t)\| \le \phi(t)\ \|u(t)\| \tag{2.1}$$

if $u$ is continuously differentiable on $[a,b]$, takes values in $\mathcal{D}(B)$, and $f(t) \equiv du/dt - Bu$ satisfies $\|f(t)\| \le \phi(t)\ \|u(t)\|$, $a \le t \le b$. $u$ is a *weak solution* of (2.1) if $u$ is continuous and for continuously differentiable functions $\psi$ with compact support in $]a,b[$ and with values in $\mathcal{D}(B^*)$, we have

$$-\int_a^b \langle u(t),\psi'(t)\rangle\, dt = \int_a^b \{\langle u(t),B^*\psi(t)\rangle + \langle f(t),\psi(t)\rangle\}dt\ ,$$

$$\|f(t)\| \le \phi(t)\ \|u(t)\|\ , \quad a \le t \le b.$$

That a strong solution of (2.1) is a weak solution follows from an integration by parts.

THEOREM 2.1. *Let* $u : [a,b] \to \mathcal{H}$ *be a weak solution of* (2.1) *where* $B$ *is symmetric. If*

$$\int_a^b \phi(t)dt \le c < 1/2\ , \tag{2.2}$$

*then the convexity inequality*

$$\|u(t)\| \le K_c\ \|u(a)\|^{\alpha}\ \|u(b)\|^{1-\alpha}\ , \tag{2.3}$$

*holds, where*

$$\alpha = \frac{b-t}{b-a}\ , \quad K_c = \left(\frac{2}{1-2c}\right)^{1/2} .$$

In particular, when $c = 1/2\sqrt{2}$, we get $K_c = (4 + 2\sqrt{2})^{1/2}$. Agmon and Nirenberg [1] proved this result for strong solutions, taking $c = 1/2\sqrt{2}$ and obtaining the constant $K_c = 2\sqrt{2}$ ($> (4 + 2\sqrt{2})^{1/2}$). This result also appears in Friedman's book [3, p.219]. Zaidman [7] extended the Agmon-Nirenberg result to weak solutions. The new features of Theorem 2.1 are (i) the result is extended to cover the case $\frac{1}{2\sqrt{2}} < c < \frac{1}{2}$, (ii) the constant $K_c$ is sharpened for each value of $c$ (including $c \le 1/2\sqrt{2}$).

By enlarging the Hilbert space $H$, we can extend $B$ to be a self-adjoint operator (cf. Sz.-Nagy [5]). Also, for $S_1$ and $S_2$ commuting self-adjoint operators (i.e., $e^{itS_1}$ and $e^{isS_2}$ commute for all real $t$ and $s$), we may extend the theorem to the case where $B$ is replaced by the (unbounded) normal operator $S_1 + iS_2$ according to the observation made in [1, p.138].

PROOF OF THEOREM 2.1. The proof follows Zaidman [7, pp. 236-244] with the following changes on pp. 242-244. We use Zaidman's notation. From

$$\|u(t)\|^2 \le \|u_1(b)\|^2 + \|u_2(a)\|^2 + 2M \int_a^b \|f(s)\|\, ds$$

(cf. [7, p.242, line 3]) we get

$$\|u(t)\|^2 \le \|u_1(b)\|^2 + \|u_1(a)\|^2 + \varepsilon M^2 + \varepsilon^{-1}\left(\int_a^b \|f(s)\|\, ds\right)^2$$

for each $\varepsilon > 0$; here $M = \sup\{\|u(s)\| : a \le s \le b\}$. This implies

$$M^2 \le \beta + \varepsilon M^2 + \varepsilon^{-1} N^2$$

where $\beta = \|u(a)\|^2 + \|u(b)\|^2$, $N = \int_a^b \|f(s)\|\, ds$. Consequently

$$M^2 \le (\beta + \varepsilon^{-1} N^2)(1 - \varepsilon)^{-1} \tag{2.4}$$

for $0 < \varepsilon < 1$. (This becomes [7, p.242, eqn. (*)] when $\varepsilon = 1/2$.) Since $u$ is a weak solution of $u' - Bu = f$ (where $\|f(t)\| \le \phi(t)\,\|u(t)\|$), it follows that $\omega_\sigma(t) \equiv e^{\sigma t}\,u(t)$ is a weak solution of $\omega' - B_\sigma\omega_\sigma = e^{\sigma t}\,f(t)$ where $B_\sigma = B - \sigma I$ (cf. [7, Lemma 4, p.242]). Letting

$$M_\sigma = \sup\,\{\|e^{\sigma t}\,u(t)\|^2 : a \le t \le b\}\ ,$$

$$B_\sigma = \|e^{\sigma a}\,u(a)\|^2 + \|e^{\sigma b}\,u(b)\|^2\ ,$$

$$N_\sigma = \int_a^b \|e^{\sigma t}\,f(t)\|\,dt\ ,$$

we have that (2.4) (applied to $\omega_\sigma$ rather than $u$) yields

$$M_\sigma^{\ 2} \le (\beta_\sigma + \varepsilon^{-1}\,N_\sigma^{\ 2})(1 - \varepsilon)^{-1} \tag{2.5}$$

for all real $\sigma$ and all $\varepsilon$, $0 < \varepsilon < 1$. But by (2.1) and (2.2),

$$\begin{aligned} N_\sigma &\le \int_a^b e^{\sigma t}\,\phi(t)\,\|u(t)\|\,dt \\ &\le \sup\,\{\|e^{\sigma s}\,u(s)\| : a \le s \le b\}\int_a^b \phi(t)\,dt \\ &\le M_\sigma\,c. \end{aligned}$$

Squaring this gives

$$N_\sigma^{\ 2} \le M_\sigma^{\ 2}\,c^2.$$

Plugging into (2.5) yields

$$M_\sigma^{\ 2} \le (\beta_\sigma + \varepsilon^{-1}\,c^2\,M_\sigma^{\ 2})(1 - \varepsilon)^{-1}$$

or

$$M_\sigma^{\ 2} \le \frac{\varepsilon\,\beta_\sigma}{\varepsilon(1-\varepsilon) - c^2} \tag{2.6}$$

provided $0 < \varepsilon < 1$ and $\varepsilon(1 - \varepsilon) > c^2$, i.e., $0 < c < 1/2$ and $|2\varepsilon - 1| < (1 - 4c^2)^{1/2}$. As in [7, pp. 243, 244], $u(a) = 0$ or $u(b) = 0$ implies $u \equiv 0$, so to prove the theorem we may suppose $u(a) \neq 0$, $u(b) \neq 0$. Choosing $\sigma = (b - a)^{-1} \log(\|u(a)\| / \|u(b)\|)$ makes

$e^{\sigma t} = (\|u(a)\| / \|u(b)\|)^{\frac{t}{b-a}}$ and $\|e^{\sigma a} u(a)\| = \|e^{\sigma b} u(b)\|$. Thus (2.6) becomes, for all $t \in [a,b]$,

$$\left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2t}{b-a}} \|u(t)\|^2 \le L \left\{ \|u(a)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2a}{b-a}} + \|u(b)\|^2 \left(\frac{\|u(a)\|}{\|u(b)\|}\right)^{\frac{2b}{b-a}} \right\}$$

$$= 2L \left(\frac{\|u(a)\|^{2b}}{\|u(b)\|^{2a}}\right)^{\frac{1}{b-a}}$$

where $L = \varepsilon(\varepsilon(1-\varepsilon) - c^2)^{-1}$. Consequently

$$\|u(t)\| \le (2L)^{1/2} \|u(a)\|^{\frac{b-t}{b-a}} \|u(b)\|^{\frac{t-a}{b-a}}$$

holds for $a \le t \le b$. Regard $g(\varepsilon) \equiv (2L)^{1/2} = \left(\dfrac{2\varepsilon}{\varepsilon(1-\varepsilon) - c^2}\right)^{1/2}$ as a function of $\varepsilon$. It is minimized when $\varepsilon = c^*$, in which case $(2L)^{1/2} = \left(\dfrac{2}{1-2c}\right)^{1/2}$. This is a legitimate choice of $\varepsilon$ since $|2\varepsilon - 1| < (1 - 4c^2)^{1/2}$ holds in this case. The proof of the theorem is now complete.

3. <u>BOUNDED SOLUTIONS</u>. Let $S_1$, $S_2$ be commuting self-adjoint operators on $H$. We study functions $u \in C^1(\mathbb{R}, H)$ ($\mathbb{R} = ]-\infty,\infty[$) which are bounded (strong) solutions of

$$du(t)/dt = (S_1 + iS_2)\,u(t)\ , \quad t \in \mathbb{R}\,. \tag{3.1}$$

LEMMA 3.1. *Let* $u$ *be a bounded solution of* (3.1). *Then* $u(t) = e^{itS_2}h$ *for all* $t \in \mathbb{R}$ *and some* $h \in \mathrm{Ker}(S_1) = \{f \in \mathcal{H} : S_1 f = 0\}$.

PROOF. Let $h = u(0)$. Then

$$u(t) = e^{tS_1}(e^{itS_2}h) = e^{itS_2}(e^{tS_1}h).$$

(Recall that $e^{tS_1}$, $e^{itS_2}$ are defined by the operational calculus associated with the spectral theorem.) Since $e^{itS_2}$ is unitary, $\|u(t)\| = \|e^{tS_1}h\|$ follows. But $\|e^{tS_1}h\|$ is bounded for $t \in \mathbb{R}$ if and only if $h \in \mathrm{Ker}(S_1)$, in which case $e^{tS_1}h = h$, and so $u(t) = e^{itS_2}h$, as advertised.

A special case occurs when

$$\mathrm{Ker}(S_1) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n\ ,$$

where $S_2$ restricted to $\mathcal{M}_j$ is a real constant $\lambda_j$ times the identity on $\mathcal{M}_j$ for $1 \le j \le n$. Then any bounded solution of (3.1) is of the form

$$u(t) = \sum_{j=1}^{n} e^{it\lambda_j} h_j \tag{3.2}$$

where $h_j$ is the orthogonal projection of $u(0)$ onto $\mathcal{M}_j$, $1 \le j \le n$. This covers the result obtained by Zaidman in [8]. More precisely, let $\{E(\theta) : \theta \in \mathbb{R}\}$ be a resolution of the identity and let

$$S_1 = \int_{-\infty}^{\infty} x(\theta)\, dE(\theta)\ , \quad S_2 = \int_{-\infty}^{\infty} y(\theta)\, dE(\theta)$$

be associated commuting self-adjoint operators, where $x$ and $y$ are continuous real functions on $\mathbb{R}$. If the zero set of $x$ is the finite set $\{\theta_1,\ldots,\theta_n\}$ then $S_2$ is $\lambda_j = y(\theta_j)$ times the identity on $M_j = (E(\theta_j^+) - E(\theta_j^-))(H)$, $1 \le j \le n$, and so any bounded solution of (3.1) is of the form (3.2) with $h_j \in M_j$, $1 \le j \le n$. This is Zaidman's result [8].

## 4. ALMOST PERIODIC SOLUTIONS.

THEOREM 4.1. *Let* $A : H \to H$ *be a bounded linear operator and let* $f : \mathbb{R} \to H$ *be almost periodic. Let* $u : \mathbb{R} \to H$ *be a bounded (i.e.* $\sup\{\|u(t)\| : t \in \mathbb{R}\} < \infty$*) strong solution of*

$$du(t)/dt = Au(t) + f(t) \qquad (t \in \mathbb{R}). \tag{4.1}$$

$$\left.\begin{array}{l}\textit{Suppose there is a finite dimensional subspace } H_1 \textit{ of } H \\ \textit{such that } H_1 \supset \{Af(s) : s \in \mathbb{R}\} \cup \{Au(0)\} \textit{ and} \\ e^{tA}(H_1) \subset H_1 \textit{ for all } t \in \mathbb{R}.\end{array}\right\} \tag{4.2}$$

*Then* $u$ *is almost periodic.*

When $H$ is finite dimensional, this is the classical Bohr-Neugebauer-Bochner theorem (cf. Amerio-Prouse [2, p.85]). When $A$ is a finite rank operator we can take $H_1$ to be the range of $A$, and Theorem 4.1 becomes the theorem of Zaidman [6] in this case.

PROOF OF THEOREM 4.1. Let $H_2 = H \ominus H_1$ be the orthogonal complement of $H_1$, and let $P_j$ be the orthogonal projection onto $H_j$, $j = 1,2$. Let $u_j(t) = P_j u(t)$, $j = 1,2$. Note that if $L$ is as upper bound for $\|u(s)\|$ $(s \in \mathbb{R})$, then for all real $t$,

$$L^2 \ge \|u(t)\|^2 = \|u_1(t)\|^2 + \|u_2(t)\|^2,$$

whence $u_1$ and $u_2$ are bounded. Also,

$$du/dt = du_1/dt + du_2/dt = Au_1 + Au_2 + P_1f + P_2f \ .$$

Applying $P_1$ to both sides gives

$$du_1/dt = P_1Au_1 + P_1Au_2 + P_1f \ . \tag{4.3}$$

The function $u$ admits the variation of parameters representation

$$\begin{aligned} u(t) &= e^{tA}u(0) + \int_0^t e^{(t-s)A} f(s)\, ds \\ &= e^{tA}u(0) + \int_0^t f(s)\, ds + \sum_{n=1}^{\infty} \int_0^t \frac{(t-s)^n}{n!} A^n f(s)\, ds \ . \end{aligned}$$

The last (summation) term belongs to $H_1$ by (4.2). Applying $P_2$ to this expression gives

$$u_2(t) = P_2 \, e^{tA}u(0) + \int_0^t P_2 \, f(s)\, ds \ ;$$

differentiating yields

$$du_2(t)/dt = P_2 \, e^{tA}Au(0) + P_2 \, f(t) = P_2 \, f(t)$$

by (4.2). Since $f$ is almost periodic and $P_2$ is bounded it follows that $du_2/dt$ is almost periodic. Since $u_2$ is bounded, $u_2$ itself is almost periodic (see [2, p.55]).

Next, by (4.3),

$$du_1(t)/dt = P_1Au_1(t) + g(t) \ , \tag{4.4}$$

where $g(t) \equiv P_1Au_2(t) + P_1f(t)$ is almost periodic. Since $u_1$ is bounded and $P_1A : H_1 \to H_1$ is linear, (4.4) is a linear system in the finite dimensional Hilbert space $H_1$ (see (4.2)). It follows from the classical Bohr-Neugebauer-Bochner theorem [2] that $u_1$ is almost periodic. Consequently $u = u_1 + u_2$ is almost periodic, and the proof is complete.

Theorem 4.1 can be easily extended to the case when A is unbounded, as follows.

THEOREM 4.2. *Let* $A : \mathcal{D}(A) \subset H \to H$ *generate a* $(C_0)$ *group of bounded linear operators* $\{T(t) : t \in \mathbb{R}\}$ *on* $H$ (cf. [4]). *Let* $u : \mathbb{R} \to H$ *be a bounded solution of* (4.1) *where* f *is almost periodic. Suppose there is a finite dimensional subspace* $H_1$ *of* $H$ *such that*

$$H_1 \supset \{(T(t) - I) f(s) : s \in \mathbb{R}, t \in \mathbb{R}\} \cup \{Au(0)\}$$

*and* $T(t)(H_1) \subset H_1$ *for all* $t \in \mathbb{R}$. *Then* u *is almost periodic.*

The proof, which differs from the proof of Theorem 4.1 only in inessential ways, is omitted.

COROLLARY 4.3. *Let* $\lambda_1, \ldots, \lambda_n$ *be eigenvalues of the linear operator* $A : \mathcal{D}(A) \subset H \to H$ *and let* $\phi_1, \ldots, \phi_n$ *be corresponding eigenvectors. Let* $H_1$ *be the span of* $\phi_1, \ldots, \phi_n$. *Then any bounded solution of* (4.1) *is almost periodic, provided* $f : \mathbb{R} \to H_1$ *is almost periodic and* $u(0) \in H_1$.

This follows immediately from Theorem 4.2.

COROLLARY 4.4. *In Corollary* 4.3 *one can omit the hypothesis that* $u(0) \in H_1$ *provided that one assumes that* A *is a compact normal operator.*

PROOF. Let $P_1$ , $P_2$ , $u_1$ , $u_2$ be as in the proof of Theorem 4.1. Applying $P_j$ to (4.1) and noting that A commutes with $P_j$ in this case gives

$$du_1(t)/dt = Au_1(t) + f(t) ,$$

$$du_2(t)/dt = Au_2(t) \qquad (t \in \mathbb{R}) . \tag{4.5}$$

$u_1$ is almost periodic by the Bohr-Neugebauer-Bochner theorem. Thus it only remains to show that $u_2$ is almost periodic. Let $B$ be the restriction of $A$ to $H_2$. $B$ is a compact normal operator, hence by the spectral theorem there is an orthonormal basis $\{\psi_m\}$ for $H_2$ and complex numbers $\mu_m \to 0$ such that

$$B\phi = \sum_{m=1}^{\infty} \mu_m \langle\phi,\psi_m\rangle \psi_m$$

for all $\phi \in H_2$. Let $Q_m$ be the orthogonal projection (in $H_2$) onto the span of $\psi_1, \ldots, \psi_m$. Let $v_m = Q_m u_2$. Then

$$dv_m/dt = Q_m\, du_2/dt = Q_m\, Au_2 = Bv_m$$

by (4.5). Also, $v_m$ is bounded (since $u_2$ is) and takes values in a finite dimensional space, whence $v_m$ is almost periodic. We *claim* that $u_2(t) = \lim_{m\to\infty} v_m(t)$ , uniformly for $t \in \mathbb{R}$ . It then follows that $u_2$ is almost periodic [2] and the proof is done. So it only remains to prove the *claim*. We have

$$\frac{d}{dt}(u_2(t) - v_m(t)) = B(u_2(t) - v_m(t)) = (B - Q_m B)(u_2(t) - v_m(t)) ,$$

therefore

$$u_2(t) - v_m(t) = \sum_{k=m+1}^{\infty} e^{t\mu_k} \langle(u_2 - v_m)(0), \psi_k\rangle \psi_k .$$

Consequently

$$\|u_2(t) - v_m(t)\|^2 = \sum_{k=m+1}^{\infty} e^{t \operatorname{Re} \mu_k} |\langle(u_2 - v_m)(0), \psi_k\rangle|^2 .$$

Since $\|u_2(t) - v_m(t)\| \leq \|u_2(t)\| \leq L < \infty$ for some $L$ and all $t \in \mathbb{R}$, it follows that for every $k$ for which $<(u_2 - v_m)(0), \psi_k> \neq 0$ for some $m$, $\mu_k$ must be purely imaginary. Therefore

$$\|u_2(t) - v_m(t)\|^2 = \sum_{k=m+1}^{\infty} |<(u_2 - v_m)(0), \psi_k>|^2$$

$$= \|(I - Q_m)u_2(0)\|^2 \to 0$$

as $n \to \infty$, uniformly for $t \in \mathbb{R}$. Q.E.D.

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