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SUBSTITUTION RELATIONS FOR LAPLACE TRANSFORMATIONS

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ABSTRACT. Additional results are obtained which center around expressions for the Laplace transforms of functions of the form $k(t)F[g(t)]$. The finite Laplace transformation is involved in a number of the formulas. Examples involving several special cases of g and k are included.

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1. INTRODUCTION.

Expressions for the Laplace transforms of certain composite functions, such as $F(t^{-1})$, $F(t^{2})$, $F(e^{t}-1)$, and $F(\sinh t)$, have been known and listed in the tables under "general formulas" for many years. In [1] a formula for $L\{k(t)\ F[g(t)]\}$ was developed and in [2] several special cases, supplementary to those in the literature, were adjoined to the list. The results contained in this paper involve both the Laplace transformation and the finite Laplace transformation (that is, the integral is over a finite interval). They do not seem to have appeared in the literature and further, although they are not difficult to obtain, do not seem to be otherwise well known.

Throughout this work we assume the form

$$f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt \tag{1.1}$$

for the Laplace transformation and we use the notations

$$f(s;(\beta,\gamma)) = L\{F(t)\ [U(t-\beta) - U(t-\gamma)]\} = \int_\beta^\gamma e^{-st} F(t)\, dt. \tag{1.2}$$

for the finite Laplace transformation. (We also allow $\gamma = \infty$.) We refer to the tables of Roberts and Kaufman [3] throughout and we use the notation [II. 3.2 (4)], for example, to refer to Part II, Section 3.2, Formula 4. The Heaviside (unit step) function is denoted by U.

2. GENERAL RESULTS.

Since our results are centered around modifications of it, we restate Theorem 1 of [1].

THEOREM 1. If (i) k, g, and the inverse function $h = g^{-1}$ are analytic, real on $(0,\infty)$, and such that $g(0) = 0$ and $g(\infty) = \infty$ (or $g(0) = \infty$ and $g(\infty) = 0$); (ii) $L\{F\} = f$ with abscissa of convergence 0; (iii) there exists a function $\Phi(s,u)$, $L\{\Phi(s,u)\} = \phi(s,p)$ with abscissa of convergence 0 and

$$\phi(s,p) = e^{-sh(p)}\ k[h(p)]\ |h'(p)|; \tag{2.1}$$

and (iv)

$$\int_0^\infty \int_0^\infty e^{-up}\ \Phi(s,u)\ F(p)\ du\ dp \tag{2.2}$$

converges absolutely for $\operatorname{Re}(s) > a$; then

$$L\{k(t)\ F[g(t)]\} = \int_0^\infty \Phi(s,u)\ f(u)\ du \tag{2.3}$$

with abscissa of convergence a.

For our first result we relax the conditions on g from those stated in Theorem 1. In connection with this we introduce the finite Laplace transformation (1.2). Now if g is strictly monotone on some subinterval of $(0,\infty)$ we have the following result.

THEOREM 2. Under the hypotheses of Theorem 1, except that now let g be monotone on (b,c) with $g(b) = \beta < \gamma = g(c)$ (or with $\beta > \gamma$), then

$$L\{k(t)\ F[g(t)]\ [U(t-b) - U(t-c)]\} = \int_0^\infty \Phi(s,u)\ f(u;(\beta,\gamma))\ du. \tag{2.4}$$

For our second result we modify Theorem 1 by the introduction of an adjustment function as follows.

THEOREM 3. Under the hypotheses of Theorem 1, except that we assume the relations

$$L\{F(p)A(p)\} = f(s;A),\quad L\{\Phi(s,u;A)\} = \phi(s,p)/A(p), \tag{2.5}$$

then

$$L\{k(t)\ F[g(t)]\} = \int_0^\infty \Phi(s,u;A)\ f(u;A)\ du. \tag{2.6}$$

The proofs of Theorems 2 and 3 follow directly the lines of proof given for Theorem 1 in [1] and hence they are omitted. Two special cases of Theorem 2 are noted.

COROLLARY 2.1. If in Theorem 2 $g(0) = \beta$, $g(\infty) = \gamma$, then

$$L\{k(t)\ F[g(t)]\} = \int_0^\infty \Phi(s,u)\ f(u;(\beta,\gamma))\ du. \tag{2.7}$$

COROLLARY 2.2. If in Theorem 2 $g(b) = 0$, $g(c) = \infty$, then

$$L\{k(t)\ F[g(t)]\ [U(t-b) - U(t-c)]\} = \int_0^\infty \Phi(s,u)\ f(u)\ du. \tag{2.8}$$

It should be noted that Theorems 2 and 3 could be combined, in which case under the joint hypotheses

$$L\{k(t)\ F[g(t)]\ [U(t-b) - U(t-c)]\} = \int_0^\infty \Phi(s,u;A)\ f(u;A;(\beta,\gamma))\ du. \tag{2.9}$$

where $f(u;A;(\beta,\gamma))$ denotes the finite transform of the product function FA over the interval (β,γ).

3. SPECIAL RESULTS.

We next turn to some examples for the illustration of those results of Section 2. Much of the computational detail is straightforward, but often lengthy, and hence is omitted. A number of substitution relations are obtained which have not appeared in tables.

EXAMPLE 1. Let $g(t) = e^{at}$, $a > 0$, $k(t) = t$. Thus from Corollary 2.1 and [II. 4.2 (2)] we have

$$L\{tF(e^{at})\} = [as\Gamma(s/a)]^{-1} \int_0^\infty u^{s/a} [\psi(s/a+1) - \log u]\, f(u;(1,\infty))\, du \tag{3.1}$$

where $f(u;(1,\infty)) = L\{F(t)\, U(t-1)\}$.

EXAMPLE 2. Let $g(t) = (t^2-a^2)^{1/2}$, $k(t) = (t^2-a^2)^{-1/2}$. From Corollary 2.2 and [II. 3.2 (46)] it follows that

$$L\{(t^2-a^2)^{-1/2} F\big((t^2-a^2)^{1/2}\big)\, U(t-a)\} = \int_s^\infty J_0\big(a(u^2-s^2)^{1/2}\big)\, f(u)\, du. \tag{3.2}$$

EXAMPLE 3. Let $g(t) = (t^2+a^2)^{1/2}$, $k(t) = (t^2+a^2)^{-1/2}$. From Corollary 2.1 and [II. 3.1 (90)] the analog to Example 2 is

$$L\{(t^2+a^2)^{-1/2}\ F\big((t^2+a^2)^{1/2}\big)\} = \int_s^\infty I_0\big(a(u^2-s^2)^{1/2}\big)\, f(u;(a,\infty))\, du. \tag{3.3}$$

EXAMPLE 4. If $g(t) = t^2 \pm a^2$, $k(t) = t^{2\nu+1}$, then [II. 3.2 (24)] along with the exponential shift can be used to obtain

$$L\{t^{2\nu+1}\ F(t^2+a^2)\} =$$

$$= 2^{-\nu-3/2}\pi^{-1/2} \int_0^\infty e^{a^2u-s^2/8u}\, u^{-\nu-1}\, D_{2\nu+1}\big(s(2u)^{-1/2}\big)\, f(u;(a^2,\infty))\, du, \tag{3.4}$$

$$L\{t^{2\nu+1}\ F(t^2-a^2)\ U(t-a)\} =$$

$$= 2^{-\nu-3/2}\pi^{-1/2} \int_0^\infty e^{-a^2u-s^2/8u} u^{-\nu-1}\, D_{2\nu+1}\big(s(2u)^{-1/2}\big)\, f(u)\, du, \tag{3.5}$$

where D_ν denotes the parabolic cylinder function.

EXAMPLE 5. If $g(t) = \cosh t$, $k(t) = e^{-at}$, from [II. 2 (99)] we have

$$L\{e^{-at}\ F(\cosh t)\} = \int_0^\infty I_{s+a}(u)\ f(u;(1,\infty))\ du. \tag{3.6}$$

EXAMPLE 6. The choice $A(p) = (ap+b)^n$ in Theorem 3 introduces an integration by parts formula; that is, where * denotes convolution, we have

$$L\{k(t)\ F[g(t)]\} =$$

$$= \int_0^\infty \left(\Phi(s,u) * [a^n\Gamma(n)]^{-1}\ u^{n-1}\ e^{-bu/a}\right)\ (-aD_u+b)^n f(u)\ du. \tag{3.7}$$

4. RATIOS OF LINEAR AND QUADRATIC EXPRESSIONS.

If we examine the general bilinear (Möbius) substitution, because of the known formula for $L\{F(ct)\}$, it is no restriction to consider only $(t-a)/(t-b)$, $(a-t)/(t-b)$, and $a/(t-b)$. A number of subcases result. In order to obtain the following results we use Theorem 2 along with [II. 3.2 (9)] and [II. 3.2 (10)], with $\nu > -1$, throughout (4.1) - (4.7).

If $b > a > 0$,

$$L\{(t-b)^{\nu-1}\ F[(t-a)/(t-b)]\ [1 - U(t-a)]\} =$$

$$= e^{-bs} \int_0^\infty e^u\ (u(b-a)/s)^{\nu/2}\ J_\nu\left(2(s(b-a)u^{1/2}\right)\ f(u;(0,a/b))\ du; \tag{4.1}$$

if $b > a$, $b > 0$,

$$L\{(t-b)^{\nu-1}\ F[(t-a)/(t-b)]\ U(t-b)\} =$$

$$= e^{-bs} \int_0^\infty e^u\ (u(b-a)/s)^{\nu/2}\ J_\nu\left(2(s(b-a)u)^{1/2}\right)\ f(u;(1,\infty))\ du; \tag{4.2}$$

if $0 > b > a$,

$$L\{(t-b)^{\nu-1} F[(t-a)/(t-b)]\} =$$

$$= e^{-bs} \int_0^\infty e^u (u(b-a)/s)^{\nu/2} J_\nu\left(2(s(b-a)u)^{1/2}\right) f(u;(1,a/b))\, du. \tag{4.3}$$

For the three corresponding cases $a > b > 0$; $a > b$, $a > 0$; and $0 > a > b$; J_ν must be replaced by I_ν, a and b interchanged in the U-functions, $(b-a)$ replaced by $(a-b)$, and the f's replaced by $f(u;(a/b,\infty))$, $f(u;(0,1))$, and $f(u;(a/b,1))$ in (4.1), (4.2), and (4.3) respectively.

If $a > b > 0$,

$$L\{(t-b)^{\nu-1} F[(a-t)/(t-b)]\ [U(t-b) - U(t-a)]\} =$$

$$= e^{-bs} \int_0^\infty e^{-u} (u(a-b)/s)^{\nu/2} J_\nu\left(2(s(a-b)u)^{1/2}\right) f(u)\, du; \tag{4.4}$$

if $a > 0$, $b < 0$,

$$L\{(t-b)^{\nu-1} F[(a-t)/(t-b)][1 - U(t-a)]\} =$$

$$= e^{-bs} \int_0^\infty e^{-u} (u(a-b)/s)^{\nu/2} J_\nu\left(2(s(a-b)u)^{1/2}\right) f(u;(0,-a/b))\, du. \tag{4.5}$$

For the corresponding cases $b > a > 0$ and $b > 0$, $a < 0$ we again change J_ν to I_ν, interchange a and b in the U-function and whenever $a-b$ appears, and in (4.5) replace $f(u;(0,-a/b))$ by $f(u;(-a/b,\infty))$.

If $a > 0$, $b > 0$,

$$L\{(t-b)^{\nu-1} F[a/(t-b)]\ U(t-b)\} =$$

$$= e^{-bs} \int_0^\infty (au/s)^{\nu/2} J_\nu\left(2(asu)^{1/2}\right) f(u)\, du; \tag{4.6}$$

if $a > 0$, $b < 0$

$$L\{(t-b)^{\nu-1} F[a/(t-b)]\} =$$

$$= e^{-bs} \int_0^\infty (au/s)^{\nu/2} J_\nu\left(2(asu)^{1/2}\right) f(u;(0,-a/b))\, du. \tag{4.7}$$

For $a < 0$, $b > 0$ modifications similar to those already discussed can be applied to (4.7).

For the ratios of quadratic functions it is again no real restriction to assume special values for some of the coefficients. In general these are messy and the inverse to $\phi(s,p)$ can not readily be obtained, hence we restrict our discussion to only a few cases. The two special cases $(a/2)t^2/(t+c)$ and $(t/2)(t+2c)/(t+c)$ have already appeared in [2]; the generalization of these to $(t^2+at+b)/(t+c)$ can be obtained.

We let $\alpha^2 = c^2-ac+b$ and for $a^2 > 4b$ we let $2\tau = -a+\sqrt{(a^2-4b)}$ in order to simplify notations. Further, we assume $c > 0$ throughout. If $\alpha^2 < 0$ and $\tau \le 0$, we have, after considerable computation,

$$L\{(t+c)^{-1-\nu} F[(t^2+at+b)/(t+c)]\} =$$

$$= e^{cs} \int_s^\infty e^{-(2c-a)u} ((u-s)/\alpha^2 u)^{\nu/2} I_\nu\left(2\alpha(u^2-su)^{1/2}\right) f(u;(b/c,\infty))\, du; \tag{4.8}$$

if $\alpha^2 > 0$ and $\tau > 0$,

$$L\{(t+c)^{-1-\nu} F[(t^2+at+b)/(t+c)]\, U(t-\tau)\} =$$

$$= e^{cs} \int_s^\infty e^{-(2c-a)u} ((u-s)/\alpha^2 u)^{\nu/2} I_\nu\left(2\alpha(u^2-su)^{1/2}\right) f(u)\, du. \tag{4.9}$$

On the other hand, if $\alpha^2 < 0$ the only alteration of the results (4.8) and (4.9) is the replacement of I_ν by J_ν. If $a^2 < 4b$, then for $c^2 > \alpha^2 > 0$, formula (4.8) is valid, but if $\alpha^2 > c^2$ we have

$$L\{(t+c)^{-1-\nu} F[(t^2+at+b)/(t+c)] U(t+c-\alpha)\} =$$

$$= e^{cs} \int_s^\infty e^{-(2c-a)u} ((u-s)/\alpha^2 u)^{\nu/2} I_\nu\left(2\alpha(u^2-su)^{1/2}\right) f(u;(\mu,\infty))\, du, \tag{4.10}$$

where $\mu = 2\alpha+a-2c$.

REFERENCES

1. BUSCHMAN, R. G. A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel, Pacific J. Math., 7 (1957), 1529-1533.

2. BUSCHMAN, R. G. Substitution formulas for Laplace transformations, Math. Comp. 22 (1968), 209-211.

3. ROBERTS, G. E. and KAUFMAN, H. Tables of Laplace Transforms, W. B. Saunders Co., Philadelphia, 1966.