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ON THE FUNCTION $\mathrm{Tr}\, e^{H+itK}$

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We prove, for two free semicircularly distributed selfadjoint elements a and b in a type II_1 von Neumann algebra with faithful trace τ, that the function $t \in \mathbb{R} \mapsto \tau(\exp(a+itb))$ is positive definite. This shows that the Bessis-Moussa-Vilani conjecture holds for large random matrices in an asymptotic sense.

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1. Introduction. Let H and K be selfadjoint $n \times n$ matrices. It is a widely known conjecture [1, 4, 8] that the function

$$t \longmapsto \mathrm{Tr}\, e^{H+itK} \tag{1.1}$$

is positive definite on $\mathbb{R}$. This means that for every $t_1, t_2, \ldots, t_k \in \mathbb{R}$, the $k \times k$ matrix

$$\left[\mathrm{Tr}\, e^{H+i(t_u - t_v)K}\right]_{u,v=1}^{k} \tag{1.2}$$

is positive semidefinite or, equivalently, that there exists a measure μ on $\mathbb{R}$ such that

$$\mathrm{Tr}\, e^{H+itK} = \int e^{itx}\, d\mu(x) \quad (t \in \mathbb{R}). \tag{1.3}$$

The function (1.1) and especially its derivatives at $t = 0$ define important quantities in quantum statistical mechanics. Proving positive definiteness would lead to interesting relations among them.

There have been several attempts to prove this conjecture but proofs have only been obtained under additional assumptions on H and K. A summary of such results has been collected in [3]. Several strengthenings of the conjecture are known to fail. For example, in the light of the Lie-Trotter formula, it would be sufficient to show that

$$\left[\mathrm{Tr}\, e^{H/m} e^{i(t_u - t_v)K/m}\right]_{u,v=1}^{k} \tag{1.4}$$

is positive semidefinite. However, this is definitely not true if $n, m, k \geq 3$, see [5].

The aim of this paper is to show that the conjecture holds asymptotically for random choices of high-dimensional matrices. More precisely, when the conjecture is true, then for any choice of selfadjoint $n \times n$ random matrices H_n and K_n, the function

$$t \longmapsto \frac{1}{n} E\left(\mathrm{Tr}\, e^{H_n + itK_n}\right) \tag{1.5}$$

is positive definite. When H_n and K_n have independent Gaussian entries and their distribution is invariant under unitary conjugation, they constitute the so-called random matrix model for semicircular selfadjoint operators a and b in a type II_1 von Neumann algebra with faithful normal tracial state τ. If H_n and K_n are independent, then Voiculescu's asymptotic freeness result about Gaussian random matrices [9] tells us that a and b are in free relation. According to asymptotic freeness, (1.5) converges to

$$t \longmapsto \tau\left(e^{a+itb}\right) \tag{1.6}$$

for every $t \in \mathbb{R}$. Hence the conjecture implies that also the function (1.6) is positive definite. The goal of the present paper is to prove this consequence of the conjecture.

For the part of free probability theory relevant to this computation we refer to [6], however, a very brief account is given below. A selfadjoint operator a is standard semicircularly distributed if

$$\tau(a^n) = \frac{1}{2\pi}\int_{-2}^{2} x^n\sqrt{4-x^2}\,dx \quad (n \in \mathbb{N}), \tag{1.7}$$

that is, if it has the same moments as the semicircular measure. For our purpose, the free relation of two standard semicircularly distributed operators a and b can be understood as a very particular rule to compute τ of any polynomial of a and b from the moments of a and b. However, there is a more efficient way in the setting of type II_1 von Neumann algebras. When x is an arbitrary element in a type II_1 von Neumann algebra with faithful normal trace τ, then there exists a unique probability measure μ_x, the Brown measure [2] of x, such that

$$\tau(g(x)) = \int_{\mathbb{C}} g(z)\,d\mu_x(z) \tag{1.8}$$

for any function g on $\mathbb{C}$, that is, analytic in a domain containing the spectrum of x. (The Brown measure extends the concept of spectral multiplicity of matrices.) The fact we really use is that, under the assumption of freeness, $a+itb$ is the so-called elliptic noncommutative distribution whose Brown measure is the two-dimensional Lebesgue measure restricted to an elliptic region of the complex plane (see [7, Theorem 4.19]).

2. Result.

We are going to prove the following theorem.

Theorem 2.1. *Let a and b be standard semicircular distributed selfadjoint operators in a von Neumann algebra with faithful normal trace τ. If a and b are in free relation with respect to τ, then $t \mapsto \tau(\exp(a+itb))$ is a positive definite function.*

Let S_α and S_β be two free semicircularly distributed selfadjoint elements with means 0 and variances $\alpha > 0$ and $\beta > 0$. We compute $\tau(\exp(S_\alpha + iS_\beta))$ using the Brown measure.

The Brown measure $\mu_{\alpha\beta}$ of $S_\alpha + iS_\beta$ is the two-dimensional Lebesgue measure restricted to the interior of the ellipse with axes $2\alpha/\sqrt{\alpha+\beta}$ and $2\beta/\sqrt{\alpha+\beta}$, see [7]. Normalized, it reads

$$\mu_{\alpha\beta}(dx\,dy) = \frac{1}{4\pi}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)dx\,dy. \tag{2.1}$$

The expectation

$$I(\alpha,\beta) := \tau\left(\exp\left(S_\alpha + iS_\beta\right)\right) \tag{2.2}$$

can now be computed using (1.8)

$$\begin{aligned} I(\alpha,\beta) &= \frac{1}{4\pi}\frac{(\alpha+\beta)}{\alpha\beta}\int_{x^2(\alpha+\beta)/4\alpha^2+y^2(\alpha+\beta)/4\beta^2\le 1} dx\, dy \exp(x+iy) \\ &= \int_{x^2+y^2\le 1} dx\, dy \exp\left(\frac{2\alpha}{\sqrt{\alpha+\beta}}x + \frac{2\beta i}{\sqrt{\alpha+\beta}}y\right) \\ &= \frac{1}{\pi}\int_0^1 r\, dr \int_0^{2\pi} d\varphi \exp\left(\frac{2\alpha r}{\sqrt{\alpha+\beta}}\cos\varphi + \frac{2\beta i r}{\sqrt{\alpha+\beta}}\sin\varphi\right). \end{aligned} \tag{2.3}$$

Next, we use for natural numbers n_1 and n_2

$$\int_0^{2\pi} d\varphi (\cos\varphi)^{2n_1}(\sin\varphi)^{2n_2} = \frac{\pi(2n_1)!(2n_2)!}{n_1!n_2!(n_1+n_2)!2^{2(n_1+n_2)-1}}. \tag{2.4}$$

Using the series expansion of the exponential and formula (2.4), we obtain

$$I(\alpha,\beta) = \sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}(\alpha-\beta)^n. \tag{2.5}$$

All other combinations of natural powers of cosine and sine give zero contributions. Therefore,

$$f(t) := \tau\left(\exp\left(S_1 + iS_{t^2}\right)\right) = \sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}(1-t^2)^n. \tag{2.6}$$

Expanding $(1-t^2)^n$ and reordering the terms in the series

$$f(t) = \sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!}\sum_{j=0}^{\infty}\frac{1}{j!(j+k+1)!}. \tag{2.7}$$

We now use the series for the modified Bessel functions I_ν

$$\mathrm{I}_\nu(z) = \sum_{m=0}^{\infty}\frac{1}{m!(\nu+m)!}\left(\frac{z}{2}\right)^{\nu+2m} \tag{2.8}$$

to get

$$f(t) = \sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!}\mathrm{I}_{k+1}(2). \tag{2.9}$$

Using the standard integral representation

$$\mathrm{I}_{k+1}(x) = \frac{1}{\Gamma(k+3/2)}\frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{k+1}\int_{-1}^{1} ds\, e^{-xs}(1-s^2)^{k+1/2} \tag{2.10}$$

for the modified Bessel functions, we find

$$f(t) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!\Gamma(k+3/2)} \int_{-1}^{1} dse^{-2s}(1-s^2)^{k+1/2}. \tag{2.11}$$

Dropping all irrelevant constants, we have to show that

$$t \longmapsto \int_{-1}^{1} dse^{-2s}(1-s^2)^{1/2} \sum_{k=0}^{\infty} \frac{(-\gamma(1-s^2)t^2)^k}{(2k+1)!} \tag{2.12}$$

is positive definite, $\gamma > 0$. This function is however just a positive superposition of functions of the type

$$t \longmapsto \frac{\sin(\delta(s)t)}{t} \tag{2.13}$$

with $\delta(s) \geq 0$. As

$$\frac{\sin(\delta t)}{t} = \frac{1}{2} \int_{-\delta}^{\delta} due^{iut}, \tag{2.14}$$

we see that the functions in (2.12) are positive definite.

We have actually computed the Fourier transform of the function (1.6). A straightforward computation yields

$$\tau(e^{a+itb}) = \frac{1}{2\pi} \int_{-2}^{2} e^{iut} \sinh\left(\sqrt{4-u^2}\right) du. \tag{2.15}$$

Clearly the Fourier transform of $\tau(e^{a+itb})$ has a compact support which coincides exactly with the (convex hull of the) spectrum of b.

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