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IMPULSIVE FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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The existence, uniqueness, and continuous dependence of a mild solution of an impulsive functional-differential evolution nonlocal Cauchy problem in general Banach spaces are studied. Methods of fixed point theorems, of a C_0 semigroup of operators and the Banach contraction theorem are applied.

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1. Introduction. In this paper, we study the existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for impulsive functional-differential evolution equation. Such problems arise in some physical applications as a natural generalization of the classical initial value problems. The results for semilinear functional-differential evolution nonlocal problem [2] are extended for the case of impulse effect. We consider the nonlocal Cauchy problem in the form

$$\begin{aligned} \dot{u}(t) &= Au(t)+f(t,u_t), \quad t\in(0,a],\ t\neq\tau_k, \\ u(\tau_k+0) &= Q_k u(\tau_k)\equiv u(\tau_k)+I_k u(\tau_k), \quad k=1,2,\dots,\kappa, \\ u(t)+\left(g(u_{t_1},\dots,u_{t_p})\right)(t) &= \phi(t), \quad t\in[-r,0], \end{aligned} \tag{1.1}$$

where $0<t_1<\cdots<t_p\le a$, $p\in\mathbb{N}$, A and I_k $(k=1,2,\dots,\kappa)$ are linear operators acting in a Banach space E; f, g, and ϕ are given functions satisfying some assumptions, $u_t(s):=u(t+s)$ for $t\in[0,a]$, $s\in[-r,0]$, $I_k u(\tau_k)=u(\tau_k+0)-u(\tau_k-0)$ and the impulsive moments τ_k are such that $0<\tau_1<\tau_2<\cdots<\tau_k<\cdots<\tau_\kappa<a$, $\kappa\in\mathbb{N}$.

Theorems about the existence, uniqueness, and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied in [1, 2, 3]. The results presented in this paper are a generalization and a continuation of some results reported in [1, 2, 3]. We consider classical impulsive functional-differential equation in the case of nonlocal condition, reduced to the classical initial functional value problem.

As usual, in the theory of impulsive differential equations [4, 5] at the points of discontinuity τ_i of the solution $t\mapsto u(t)$ we assume that $u(\tau_i)\equiv u(\tau_i-0)$. It is clear that, in general, the derivatives $\dot{u}(\tau_i)$ do not exist. On the other hand, according to the first equality of (1.1) there exist the limits $\dot{u}(\tau_i\mp 0)$. According to the above convention, we assume $\dot{u}(\tau_i)\equiv\dot{u}(\tau_i-0)$.

Throughout, we assume that E is a Banach space with norm $\|\cdot\|$, A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on E, $D(A)$ is the domain of A, and

$$M := \sup_{t\in[0,a]} \left\{ \|T(t)\|_{BL(E,E)} \right\}. \tag{1.2}$$

Let $f : [0,a]\times C([-r,0],E) \to E$. Introduce the following assumptions:

(H1) for every $w \in C([-r,a],E)$ and $t \in [0,a]$, $f(\cdot,w_t) \in C([0,a],E)$;

(H2) there exists a constant $L > 0$ such that

$$\begin{gathered} \|f(t,w_t) - f(t,\tilde{w}_t)\|_E \\ \leq L_1\|w-\tilde{w}\|_{C([-r,t],E)} \quad \text{for } w,\tilde{w} \in C([-r,a],E),\ t\in[0,a], \\ \|I_k v\|_E \leq L_2\|v\|_E \quad \text{for } v \in E,\ k = 1,2,\ldots,\kappa, \\ L = \max\{L_1,L_2\}. \end{gathered} \tag{1.3}$$

Let $g : [C([-r,0],E)]^p \to C([-r,0],E)$. Then we have the following assumptions:

(H3) there exists a constant $K > 0$ such that

$$\|(g(w_{t_1},\ldots,w_{t_p}))(t) - (g(\tilde{w}_{t_1},\ldots,\tilde{w}_{t_p}))(t)\| \leq K\|w-\tilde{w}\|_{C([-r,a],E)} \tag{1.4}$$

for $w,\tilde{w} \in C([-r,a],E)$, $t \in [-r,0]$;

(H4) assume that $\phi \in C([-r,0],E)$.

A function $u \in C([-r,a],E)$ satisfying the following conditions:

$$\begin{gathered} u(t) = T(t)\phi(0) - T(t)[(g(u_{t_1},\ldots,u_{t_p}))(0)] \\ + \int_0^t T(t-s)f(s,u_s)ds + \sum_{0<\tau_k<t} T(t-\tau_k)I_k u(\tau_k),\ t\in[0,a], \\ u(t) + (g(u_{t_1},\ldots,u_{t_p}))(t) = \phi(t), \quad t\in[-r,0), \end{gathered} \tag{1.5}$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1).

2. Existence and uniqueness of a mild solution

Theorem 2.1. *Suppose that assumptions (H1)–(H4) are satisfied and*

$$M[K + L(a+1)] < 1. \tag{2.1}$$

Then the impulsive nonlocal Cauchy problem (1.1) has a unique mild solution.

Proof. The mild solution of the impulsive system (1.1) with nonlocal condition can be written in the form

$$u(t;\phi) = (Fu)(t), \tag{2.2}$$

where

$$(Fw)(t) := \begin{cases} \phi(t) - (g(w_{t_1},\dots,w_{t_p}))(t), \quad t \in [-r,0), \\ T(t)\phi(0) - T(t)[(g(w_{t_1},\dots,w_{t_p}))(0)] \\ \quad + \int_0^t T(t-s)f(s,w_s)ds + \sum_{0<\tau_k<t} T(t-\tau_k)I_k w(\tau_k), \quad t \in [0,a], \end{cases} \tag{2.3}$$

such that $w \in C([-r,a],E)$ and $F : C([-r,a],E) \to C([-r,a],E)$. Now, we show that F is a contraction mapping on $C([-r,a],E)$. Therefore,

$$(Fw)(t) - (F\tilde{w})(t) := \begin{cases} (g(\tilde{w}_{t_1},\dots,\tilde{w}_{t_p}))(t) - (g(w_{t_1},\dots,w_{t_p}))(t) \\ \qquad \text{for } w,\tilde{w} \in C([-r,a],E),\ t \in [-r,0), \\ T(t)[(g(\tilde{w}_{t_1},\dots,\tilde{w}_{t_p}))(0) - (g(w_{t_1},\dots,w_{t_p}))(0)] \\ \quad + \int_0^t T(t-s)[f(s,w_s) - f(s,\tilde{w}_s)]ds \\ \quad + \sum_{0<\tau_k<t} T(t-\tau_k)[I_k w(\tau_k) - I_k\tilde{w}(\tau_k)] \\ \qquad \text{for } w,\tilde{w} \in C([-r,a],E),\ t \in [0,a]. \end{cases} \tag{2.4}$$

From (2.4), we have

$$\begin{aligned} ||(Fw)(t) - (F\tilde{w})(t)|| &\le ||T(t)|| \cdot ||(g(\tilde{w}_{t_1},\dots,\tilde{w}_{t_p}))(0) - (g(w_{t_1},\dots,w_{t_p}))(0)|| \\ &\quad + \int_0^t ||T(t-s)|| \cdot ||f(s,w_s) - f(s,\tilde{w}_s)||ds \\ &\quad + \sum_{0<\tau_k<t} ||T(t-\tau_k)|| \cdot ||I_k w(\tau_k) - I_k\tilde{w}(\tau_k)|| \end{aligned} \tag{2.5}$$

for $w,\tilde{w} \in C([-r,a],E)$, $t \in [0,a]$. Because of (2.5), in view of (1.2), and applying assumptions (H1)–(H4) we obtain

$$\begin{aligned} ||(Fw)(t) - (F\tilde{w})(t)|| &\le MK||w-\tilde{w}||_{C([-r,a],E)} \\ &\quad + ML_1\int_0^t ||w-\tilde{w}||_{C([-r,a],E)}ds + ML_2||w(\tau_k) - \tilde{w}(\tau_k)||_E \\ &\le (MK + MaL_1 + ML_2)||w-\tilde{w}||_{C([-r,a],E)} \\ &\le M[K + L(a+1)] \cdot ||w-\tilde{w}||_{C([-r,a],E)} \end{aligned} \tag{2.6}$$

for $w,\tilde{w} \in C([-r,a],E)$, $t \in [0,a]$, which implies that

$$||Fw - F\tilde{w}||_{C([-r,a],E)} \le \beta||w-\tilde{w}||_{C([-r,a],E)}, \quad w,\tilde{w} \in C([-r,a],E), \tag{2.7}$$

where $\beta := M[K + L(a+1)]$. The operator F satisfies all the assumptions of the Banach contraction theorem, and therefore, in the space $C([-r,a],E)$ there is only one fixed point of F and this is the mild solution of the nonlocal Cauchy problem with impulse effect. This completes the proof of the theorem. □

3. Continuous dependence of a mild solution

Theorem 3.1. *Suppose that the functions f, g, and $I_k(u)$, $k = 1, 2, \dots, \kappa$, satisfy the assumptions (H1)–(H4) and $M[K + L(a+1)] < 1$. Then, for each $\phi_1, \phi_2 \in C([-r,a],E)$, and for the corresponding mild solutions u_1, u_2 of the problems,*

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t,u_t), \quad t \in (0,a],\ t \neq \tau_k, \\ u(\tau_k + 0) &= Q_k u(\tau_k) \equiv u(\tau_k) + I_k u(\tau_k), \quad k = 1,2,\dots,\kappa, \\ u(t) + (g(u_{t_1},\dots,u_{t_p}))(t) &= \phi_i(t) \quad (i = 1,2),\ t \in [-r,0], \end{aligned} \tag{3.1}$$

the following inequality holds:

$$||u_1 - u_2||_{C([-r,a],E)} \le Me^{aML}(1+ML)^{\kappa}\Big\{||\phi_1 - \phi_2||_{C([-r,0],E)} + K||u_1 - u_2||_{C([-r,a],E)}\Big\}. \tag{3.2}$$

Additionally, if

$$K < \frac{e^{-aML}(1+ML)^{-\kappa}}{M}, \tag{3.3}$$

then

$$||u_1 - u_2||_{C([-r,a],E)} \le \frac{Me^{aML}(1+ML)^{\kappa}}{1 - KMe^{aML}(1+ML)^{\kappa}}||\phi_1 - \phi_2||_{C([-r,0],E)}. \tag{3.4}$$

Proof. Assume that $\phi_i \in C([-r,0],E)$ $(i = 1,2)$ are arbitrary functions and let u_i $(i = 1,2)$ be the mild solutions of problem (3.1). Then

$$\begin{aligned} u_1(t) - u_2(t) = T(t)&[\phi_1(0) - \phi_2(0)] \\ &- T(t)\{[g((u_1)_{t_1},\dots,(u_1)_{t_p})](0) - [g((u_2)_{t_1},\dots,(u_2)_{t_p})](0)\} \\ &+ \int_0^t T(t-s)[f(s,(u_1)_s) - f(s,(u_2)_s)]ds \\ &+ \sum_{0<\tau_k<t} T(t-\tau_k)[I_k u_1(\tau_k) - I_k u_2(\tau_k)] \end{aligned} \tag{3.5}$$

for $t \in [0,a]$ and

$$u_1(t) - u_2(t) = \phi_1(t) - \phi_2(t) - \{[g((u_2)_{t_1},\dots,(u_2)_{t_p})](t) - [g((u_1)_{t_1},\dots,(u_1)_{t_p})](t)\} \tag{3.6}$$

for $t \in [-r,0)$. From (3.5), (1.2), and using (H2) we get

$$\begin{aligned} ||u_1(\xi) - u_2(\xi)|| \le\ & M||\phi_1 - \phi_2||_{C([-r,0],E)} + MK||u_1 - u_2||_{C([-r,a],E)} \\ & + ML_1\int_0^{\xi}||u_1 - u_2||_{C([-r,s],E)}ds + ML_2\sum_{0<\tau_k<\xi}||u_1(\tau_k) - u_2(\tau_k)||_E \\ \le\ & M||\phi_1 - \phi_2||_{C([-r,0],E)} + MK||u_1 - u_2||_{C([-r,a],E)} \\ & + ML_1\int_0^{t}||u_1 - u_2||_{C([-r,s],E)}ds + ML_2\sum_{0<\tau_k<t}||u_1(\tau_k) - u_2(\tau_k)||_E \end{aligned} \tag{3.7}$$

for $0 \le \xi \le t \le a$. With this result, by virtue of (H3) it follows that

$$\begin{aligned}\sup_{\xi\in[0,t]} &\|u_1(\xi)-u_2(\xi)\| \\ &\le M\|\phi_1-\phi_2\|_{C([-r,0],E)} + MK\|u_1-u_2\|_{C([-r,a],E)} \\ &\quad + ML_1\int_0^t \|u_1-u_2\|_{C([-r,s],E)}ds + ML_2\sum_{0<\tau_k<t}\|u_1(\tau_k)-u_2(\tau_k)\|_E\end{aligned} \tag{3.8}$$

for $t \in [0,a]$. At the same time, by (3.6) and (H3) we have

$$\|u_1(t)-u_2(t)\| \le M\|\phi_1-\phi_2\|_{C([-r,0],E)} + MK\|u_1-u_2\|_{C([-r,a],E)} \tag{3.9}$$

for $t \in [-r,0)$. Formulas (3.8) and (3.9) imply that

$$\begin{aligned}\|u_1(t)-u_2(t)\| \le{}& M\|\phi_1-\phi_2\|_{C([-r,0],E)} + MK\|u_1-u_2\|_{C([-r,a],E)} \\ &+ML\left\{\int_0^t \|u_1-u_2\|_{C([-r,s],E)}ds + \sum_{0<\tau_k<t}\|u_1(\tau_k)-u_2(\tau_k)\|_E\right\}.\end{aligned} \tag{3.10}$$

Applying Gronwall's inequality for discontinuous functions (see [5]), from (3.10) it follows that

$$\begin{aligned}\|u_1(t)-u_2(t)\|_{C([-r,a],E)} \le{}& \Big\{M\|\phi_1-\phi_2\|_{C([-r,0],E)} \\ &+ MK\|u_1-u_2\|_{C([-r,a],E)}\Big\}e^{aML}(1+ML)^{\kappa}\end{aligned} \tag{3.11}$$

and therefore, (3.2) holds. Inequality (3.4) is a consequence of (3.2). This completes the proof of the theorem. □

Remark 3.2. If $K = \kappa = 0$, then (3.2) is reduced to the classical inequality

$$\|u_1(t)-u_2(t)\|_{C([-r,a],E)} \le Me^{aML}\|\phi_1-\phi_2\|_{C([-r,0],E)}, \tag{3.12}$$

which is characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

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References

[1] H. Akça and V. Covachev, *Periodic solutions of impulsive systems with delay*, Funct. Differ. Equ. **5** (1998), no. 3-4, 275–286.

[2] L. Byszewski and H. Akça, *On a mild solution of a semilinear functional-differential evolution nonlocal problem*, J. Appl. Math. Stochastic Anal. **10** (1997), no. 3, 265–271.

[3] ———, *Existence of solutions of a semilinear functional-differential evolution nonlocal problem*, Nonlinear Anal. **34** (1998), no. 1, 65–72.

[4] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Modern Applied Mathematics, vol. 6, World Scientific Publishing, New Jersey, 1989.

[5] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientific Publishing, New Jersey, 1995.

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