IJMMS 29:5 (2002) 285–290
PII. S0161171202006890
http://ijmms.hindawi.com

COEFFICIENT INEQUALITIES FOR CERTAIN ANALYTIC FUNCTIONS

JUNICHI NISHIWAKI and SHIGEYOSHI OWA

Received 1 March 2001

For real α $(\alpha > 1)$, we introduce subclasses $M(\alpha)$ and $N(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in U. The object of the present paper is to consider the coefficient inequalities for functions $f(z)$ to be in the classes $M(\alpha)$ and $N(\alpha)$. Further, the bounds of α for functions $f(z)$ to be starlike in U are considered.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $M(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \quad (z \in U) \tag{1.2}$$

for some α $(\alpha > 1)$. And let $N(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \quad (z \in U) \tag{1.3}$$

for some α $(\alpha > 1)$. Then, we see that $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$. We give examples of functions $f(z)$ in the classes $M(\alpha)$ and $N(\alpha)$.

Remark 1.1. For $1 < \alpha \leq 4/3$, the classes $M(\alpha)$ and $N(\alpha)$ were introduced by Uralegaddi et al. [2].

Example 1.2. (i) $f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha)$.
(ii) $g(z) = (1/(2\alpha - 1))\{1 - (1-z)^{2\alpha-1}\} \in N(\alpha)$.

Proof. Since $f(z) \in M(\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha, \tag{1.4}$$

we can write

$$\frac{\alpha - zf'(z)/f(z)}{\alpha - 1} = \frac{1+z}{1-z}, \tag{1.5}$$

which is equivalent to

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha-1)}{1-z}. \tag{1.6}$$

Integrating both sides of the above equality, we have

$$f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha). \tag{1.7}$$

Next, since $g(z) \in N(\alpha)$ if and only if $zg'(z) \in M(\alpha)$,

$$zg'(z) = z(1-z)^{2(\alpha-1)}. \tag{1.8}$$

For function $g(z) \in N(\alpha)$, it follows that

$$g(z) = -\frac{1}{2\alpha-1}(1-z)^{2\alpha-1} + \frac{1}{2\alpha-1} = \frac{1}{2\alpha-1}\{1-(1-z)^{2\alpha-1}\} \in N(\alpha). \tag{1.9}$$

□

2. Coefficient inequalities for the classes $M(\alpha)$ and $N(\alpha)$. We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2.1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \le 2(\alpha-1) \tag{2.1}$$

for some α $(\alpha > 1)$, then $f(z) \in M(\alpha)$.

Proof. Suppose that

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \le 2(\alpha-1) \tag{2.2}$$

for $f(z) \in A$.

It suffices to show that

$$\left| \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) - (2\alpha-1)} \right| < 1 \quad (z \in U). \tag{2.3}$$

We have

$$\begin{aligned} \left| \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) - (2\alpha-1)} \right| &\le \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{2(\alpha-1) - \sum_{n=2}^{\infty}|n-2\alpha+1||a_n||z|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{2(\alpha-1) - \sum_{n=2}^{\infty}|n-2\alpha+1||a_n|}. \end{aligned} \tag{2.4}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty} (n-1)|a_n| \le 2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1||a_n| \tag{2.5}$$

which is equivalent to condition (2.1). This completes the proof of the theorem. □

By using Theorem 2.1, we have the following corollary.

Corollary 2.2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n\{(n-1)+|n-2\alpha+1|\}|a_n| \le 2(\alpha-1) \tag{2.6}$$

for some α $(\alpha > 1)$, then $f(z) \in N(\alpha)$.

Proof. From $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$, replacing a_n by na_n in Theorem 2.1 we have the corollary. □

In view of Theorem 2.1 and Corollary 2.2, if $1 < \alpha \le 3/2$, then $n-2\alpha+1 \ge 0$ for all $n \ge 2$. Thus we have the following corollary.

Corollary 2.3. (i) *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \le \alpha-1 \tag{2.7}$$

for some α $(1 < \alpha \le 3/2)$, then $f(z) \in M(\alpha)$.

(ii) *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le \alpha-1 \tag{2.8}$$

for some α $(1 < \alpha \le 3/2)$, then $f(z) \in N(\alpha)$.

3. Starlikeness for functions in $M(\alpha)$ and $N(\alpha)$.

By Silverman [1], we know that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n|a_n| \le 1, \tag{3.1}$$

then $f(z) \in S^*$, where S^* denotes the subclass of A consisting of all univalent and starlike functions $f(z)$ in U. Thus we have the following theorem.

Theorem 3.1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \le \alpha-1 \tag{3.2}$$

for some α $(1 < \alpha \le 4/3)$, then $f(z) \in S^* \cap M(\alpha)$, therefore, $f(z)$ is starlike in U. Further, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \le \alpha-1 \tag{3.3}$$

for some α $(1 < \alpha \le 3/2)$, then $f(z) \in S^* \cap N(\alpha)$, therefore, $f(z)$ is starlike in U.

Proof. Consider α such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1}|a_n| \leq 1. \tag{3.4}$$

Then we have $f(z) \in S^* \cap M(\alpha)$ by means of Theorem 2.1. This inequality holds true if

$$n \leq \frac{n-\alpha}{\alpha-1} \quad (n = 2,3,4,\ldots). \tag{3.5}$$

Therefore, we have

$$1 < \alpha \leq 2 - \frac{2}{n+1} \quad (n = 2,3,4,\ldots), \tag{3.6}$$

which shows that $1 < \alpha \leq 4/3$. Next, considering α such that

$$\sum_{n=2}^{\infty} n|a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha-1}|a_n| \leq 1, \tag{3.7}$$

we have

$$n \leq \frac{n(n-\alpha)}{\alpha-1} \quad (n = 2,3,4,\ldots), \tag{3.8}$$

which is equivalent to

$$1 < \alpha \leq \frac{n+1}{2} \quad (n = 2,3,4,\ldots). \tag{3.9}$$

This implies that $1 < \alpha \leq 3/2$. □

Finally, by virtue of the result for convex functions by Silverman [1], we have, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1, \tag{3.10}$$

then $f(z) \in K$, where K denotes the subclass of A consisting of all univalent and convex functions $f(z)$ in U. Using the same method as in the proof of Theorem 3.1, we derive the following theorem.

Theorem 3.2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \alpha - 1 \tag{3.11}$$

for some α $(1 < \alpha \leq 4/3)$, then $f(z) \in K \cap N(\alpha)$, therefore, $f(z)$ is convex in U.

4. Bounds of α for starlikeness. Note that the sufficient condition for $f(z)$ to be in the class $M(\alpha)$ is given by

$$\sum_{n=2}^{\infty} \{(n-1)+|n-2\alpha+1|\}|a_n| \leq 2(\alpha-1). \tag{4.1}$$

Since, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n|a_n| \leq 1, \tag{4.2}$$

then $f(z) \in S^*$ (cf. [1]). It is interesting to find the bounds of α for starlikeness of $f(z) \in M(\alpha)$. To do this, we have to consider the following inequality:

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1}{2(\alpha-1)} \sum_{n=2}^{\infty} \{(n-1)+|n-2\alpha+1|\}|a_n| \leq 1 \tag{4.3}$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n-2\alpha+1|+(3-2\alpha)n\}|a_n| \geq 0. \tag{4.4}$$

We define

$$F(n) = |n-2\alpha+1|+(3-2\alpha)n \quad (n \geq 2). \tag{4.5}$$

Then, if $F(n)$ satisfies

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0, \tag{4.6}$$

then $f(z)$ belongs to S^*.

Theorem 4.1. *Let $f(z) \in A$ satisfy*

$$\sum_{n=2}^{\infty} \{(n-1)+|n-2\alpha+1|\}|a_n| \leq 2(\alpha-1) \tag{4.7}$$

for some $\alpha > 1$. Further, let δ_k be defined by

$$\delta_k = \sum_{n=k}^{\infty} F(n)|a_n|. \tag{4.8}$$

Then,

(i) *if $1 < \alpha \leq 3/2$, then $f(z) \in S^*$,*
(ii) *if $3/2 \leq \alpha \leq \min(13/8, (3+\delta_3)/2)$, then $f(z) \in S^*$,*
(iii) *if $8/3 \leq \alpha \leq \min(17/10, (12-\delta_4+\sqrt{\delta_4^2+48\delta_4+48})/12)$, then $f(z) \in S^*$.*

Proof. For $1 < \alpha \leq 3/2$, we know that

$$n-2\alpha+1 \geq 3-2\alpha \geq 0 \quad (n \geq 2), \tag{4.9}$$

that is, $F(n) \geq 0$ $(n \geq 2)$. Therefore, we have

$$\sum_{n=2}^{\infty} F(n)|a_n| \geq 0. \tag{4.10}$$

If $3/2 \leq \alpha \leq 13/8$, then $F(2) = 3-2\alpha \leq 0$ and

$$F(n) = 2n(2-\alpha)+1-2\alpha \geq 13-8\alpha \geq 0 \tag{4.11}$$

for $n \geq 3$. Further, we know that

$$|a_n| \leq \frac{2(\alpha-1)}{(n-1)+|n-2\alpha+1|} \quad (n \geq 2), \tag{4.12}$$

then $|a_2| \leq 1$. Therefore, we obtain that

$$\sum_{n=2}^{\infty} F(n)|a_n| = F(2)|a_2| + \sum_{n=3}^{\infty} F(n)|a_n| \geq 3-2\alpha+\delta_3 \geq 0 \tag{4.13}$$

for

$$\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right). \tag{4.14}$$

Furthermore, if $13/8 \leq \alpha \leq 17/10$, then

$$\begin{aligned} F(2) &= 3-2\alpha \leq 0, \\ F(3) &= |4-2\alpha| + 3(3-2\alpha) = 13-8\alpha \leq 0, \\ F(n) &= |n-2\alpha+1| + (3-2\alpha)n = 4n+1-2(n+1)\alpha \geq \frac{3(n-4)}{5} \geq 0 \end{aligned} \tag{4.15}$$

for $n \geq 4$. Noting that $|a_2| \leq 1$ and $|a_3| \leq (\alpha-1)/(3-\alpha)$, we conclude that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n)|a_n| &= F(2)|a_2| + F(3)|a_3| + \sum_{n=4}^{\infty} F(n)|a_n| \\ &\geq (3-2\alpha) + (13-8\alpha)\frac{\alpha-1}{3-\alpha} + \delta_4 \geq 0, \end{aligned} \tag{4.16}$$

for α that satisfies

$$6\alpha^2 - (12-\delta_4)\alpha + 4 - 3\delta_4 \leq 0. \tag{4.17}$$

This shows that

$$\frac{8}{3} \leq \alpha \leq \min\left(\frac{17}{10}, \frac{12-\delta_4+\sqrt{\delta_4^2+48\delta_4+48}}{12}\right). \tag{4.18}$$

This completes the proof of Theorem 4.1. □

Finally, by virtue of Theorem 4.1, we may suppose that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{(n-1)+|n-2\alpha+1|\}|a_n| \leq 2(\alpha-1) \tag{4.19}$$

for some $1 < \alpha < 2$, then $f(z) \in S^*$.

References

[1] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[2] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25** (1994), no. 3, 225–230.

Junichi Nishiwaki and Shigeyoshi Owa: Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan