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ON THE OSCILLATION OF FIRST-ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH REAL COEFFICIENTS

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We prove sufficient conditions for the oscillation of all solutions of a scalar first-order neutral delay differential equation $\dot{x}(t)-c\dot{x}(t-\tau)+\sum_{i=1}^{n} p_i x(t-\sigma_i)=0$ for all $0<c<1$, $\tau,\sigma_i>0$, and $p_i\in\mathbb{R}$, $i=1,2,\ldots,n$.

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1. Introduction. The theory of neutral delay differential equations presents complications and the results which are true for neutral differential equations may not be true for nonneutral differential equations. Besides its theoretical interest, the study of oscillatory behaviour of solutions of neutral delay differential equations has some importance in applications. Neutral delay differential equations appear in networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar and also in population dynamics (see Gopalsamy [5], Györi and Ladas [7], Driver [4], Hale [8], Brayton and Willoughby [3], Agwo [1], and the references therein).

In fact, Zahariev and Baĭnov [11] seems to be the first paper dealing with oscillation of neutral equations. A systematic development of oscillation theory of neutral equations was initiated by Ladas and Sficas [10].

Ladas and Schultz [9] obtained a necessary and sufficient condition for oscillation of all solutions of the neutral delay differential equation

$$\dot{x}(t)+c\dot{x}(t-\tau)+qx(t-\sigma)=0, \tag{1.1}$$

where c, q, τ, and σ are real numbers. It was proved that all solutions of (1.1) are oscillatory if and only if the characteristic equation

$$F(\lambda)\equiv\lambda+c\lambda e^{-\lambda\tau}+qe^{-\lambda\sigma}=0 \tag{1.2}$$

has no real roots.

Also, it was proved for the scalar first-order neutral delay differential equation

$$\dot{x}(t)+c\dot{x}(t-\tau)+\sum_{i=1}^{n} p_i x(t-\sigma_i)=0, \tag{1.3}$$

where $c\in\mathbb{R}$, $\tau,\sigma_i\geq 0$, and $p_i>0$ for all $i=1,2,\ldots,n$, that all solutions are oscillatory if and only if the characteristic equation

$$F(\lambda) \equiv \lambda + c\lambda e^{-\lambda\tau} + \sum_{i=1}^{n} p_i e^{-\lambda\sigma_i} = 0 \tag{1.4}$$

has no real roots. This result was generalized by Arino and Györi in [2] for $p_i \in \mathbb{R}$.

In [6], Gopalsamy and Zhang proved that, if

(1) $0 < c < 1$,

(2) $\tau \geq 0$, $\sigma > 0$, $p \geq 0$,

(3) $pe\sigma > 1 - c(1 + \tau p/(1-c))$,

then every solution of

$$\dot{x}(t) - c\dot{x}(t-\tau) + px(t-\sigma) = 0 \tag{1.5}$$

is oscillatory.

In this paper, we extend the last result for a scalar first-order neutral delay differential equation in the form

$$\dot{x}(t) - c\dot{x}(t-\tau) + \sum_{i=1}^{n} p_i x(t-\sigma_i) = 0 \tag{1.6}$$

for all $0 < c < 1$, $\tau, \sigma_i \geq 0$, and $p_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Let $\gamma = \max\{t, \sigma_1, \sigma_2, \ldots, \sigma_n\}$ and let $t_1 \geq t_0$. By a solution of (1.6) on $[t_1, \infty)$ we mean a function $x(t) \in \mathbf{C}([t_1 - \gamma, t_1], \mathbb{R})$ such that $x(t) - cx(t-\tau)$ is continuously differentiable and (1.6) is satisfied for $t \geq t_1$.

As it is customary, a solution is called oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory.

2. The main result.

Consider (1.6) and assume that $p_{k_i} \geq 0$ for all $i = 1, 2, \ldots, \ell$ and $p_{m_j} < 0$ for all $j = 1, 2, \ldots, r$ with $\ell + r = n$. Let $q_{m_j} = -p_{m_j}$, $j = 1, 2, \ldots, r$, then (1.6) takes the form

$$\dot{x}(t) - c\dot{x}(t-\tau) + \sum_{i=1}^{\ell} p_{k_i} x(t-\tau_{k_i}) - \sum_{j=1}^{r} q_{m_j} x(t-\sigma_{m_j}) = 0 \tag{2.1}$$

or simply

$$\dot{x}(t) - c\dot{x}(t-\tau) + \sum_{i=1}^{\ell} p_i x(t-\tau_i) - \sum_{j=1}^{r} q_j x(t-\sigma_j) = 0, \tag{2.2}$$

where $0 < c < 1$, $\tau, \sigma_i, p_i \geq 0$, and $\tau_i, q_j > 0$ for all $i = 1, 2, \ldots, \ell$ and all $j = 1, 2, \ldots, r$ with $\ell + r = n$.

Theorem 2.1. *Consider the neutral delay differential equation (2.2). If*

(i) $\ell p_i > \sum_{j=1}^{r} q_j$ *for all* $i = 1, 2, \ldots, \ell$,

(ii) $\sum_{i=1}^{\ell}(1 - c - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)) \geq 0$,

(iii) $(e + c\tau/(1-c))\sum_{i=1}^{\ell}(\ell p_i - \sum_{j=1}^{r} q_j)\tau_i > \sum_{i=1}^{\ell}((1-c) - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j))$,

then all solutions of (2.2) are oscillatory.

Proof. The characteristic equation of the neutral delay differential equation (2.2) is

$$F(\lambda) \equiv \lambda - c\lambda e^{-\lambda\tau} + \sum_{i=1}^{\ell} p_i e^{-\lambda\tau_i} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j} = 0. \tag{2.3}$$

Assume that (2.2) has a nonoscillatory solution, then the characteristic equation (2.3) has a real root λ_0, that is,

$$F(\lambda_0) \equiv \lambda_0 - c\lambda_0 e^{-\lambda_0\tau} + \sum_{i=1}^{\ell} p_i e^{-\lambda_0\tau_i} - \sum_{i=1}^{r} q_j e^{-\lambda_0\sigma_j} = 0. \tag{2.4}$$

But for all $\lambda \in \mathbb{R}$, one can write

$$\begin{aligned} &\lambda\left(1 - ce^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\int_0^{\tau_i-\sigma_j} e^{-\lambda s}\,ds\right) \\ &\quad = \lambda - \lambda c e^{-\lambda\tau} + \sum_{j=1}^{r} q_j\left(e^{-\lambda(\tau_i-\sigma_j)} - 1\right)e^{-\lambda\sigma_j} \\ &\quad = \lambda - \lambda c e^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j} + e^{-\lambda\tau_i}\sum_{j=1}^{r} q_j \end{aligned} \tag{2.5}$$

for all $i = 1,2,\dots,\ell$ and then

$$\begin{aligned} &\sum_{i=1}^{\ell}\lambda\left(1 - ce^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\int_0^{\tau_i-\sigma_j} e^{-\lambda s}\,ds\right) \\ &\quad = \ell\left(\lambda - \lambda c e^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\right) + \left(\sum_{i=1}^{\ell} e^{-\lambda\tau_i}\right)\left(\sum_{j=1}^{r} q_j\right). \end{aligned} \tag{2.6}$$

From (2.3) and (2.6), one can write

$$F(\lambda) = \frac{1}{\ell}\sum_{i=1}^{\ell}\lambda\left(1 - ce^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\int_0^{\tau_i-\sigma_j} e^{-\lambda s}\,ds\right) + \sum_{i=1}^{\ell}\left(p_i - \sum_{j=1}^{r} q_j\right)e^{-\lambda\tau_i} \tag{2.7}$$

for all $\lambda \geq 0$, we have

$$F(\lambda) > \frac{1}{\ell}\left\{\sum_{i=1}^{\ell}\lambda\left(1 - c - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right) + \sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right)e^{-\lambda\tau_i}\right\}. \tag{2.8}$$

Since $\ell p_i > \sum_{j=1}^{r} q_j$ for all $i = 1,2,\dots,\ell$ and $\sum_{i=1}^{\ell}(1 - c - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)) \geq 0$, it follows that $F(\lambda) > 0$ and consequently $F(\lambda)$ has no positive or zero real roots.

From (2.7), we have

$$\ell F(\lambda) = \lambda\left\{\sum_{i=1}^{\ell}\left(1 - ce^{-\lambda\tau} - \sum_{j=1}^{r} q_j e^{-\lambda\sigma_j}\int_0^{\tau_i-\sigma_j} e^{-\lambda s}\,ds\right) + \frac{1}{\lambda}\sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right)e^{-\lambda\tau_i}\right\}. \tag{2.9}$$

In order that $F(\lambda)$ has no roots for all $\lambda<0$, we prove that $F(\lambda)>0$ for all $\lambda<0$ and consequently it is enough to prove that

$$\sum_{i=1}^{\ell}\left(1-ce^{-\lambda\tau}-\sum_{j=1}^{r}q_je^{-\lambda\sigma_j}\int_0^{\tau_i-\sigma_j}e^{-\lambda s}\,ds\right)<-\frac{1}{\lambda}\sum_{i=1}^{\ell}\left(\ell p_i-\sum_{j=1}^{r}q_j\right)e^{-\lambda\tau_i}. \tag{2.10}$$

Assume that $-\lambda=\mu$ and put

$$\begin{aligned} f_1(\mu)&=\sum_{i=1}^{\ell}\left(1-ce^{\mu\tau}-\sum_{j=1}^{r}q_je^{\mu\sigma_j}\int_0^{\tau_i-\sigma_j}e^{\mu s}\,ds\right),\\ f_2(\mu)&=\sum_{i=1}^{\ell}\left(\ell p_i-\sum_{j=1}^{r}q_j\right)\frac{e^{\mu\tau_i}}{\mu}. \end{aligned} \tag{2.11}$$

Since $\ell p_i>\sum_{j=1}^{r}q_j$ for all $i=1,2,\ldots,\ell$, then $f_2(\mu)>e\sum_{i=1}^{\ell}(\ell p_i-\sum_{j=1}^{r}q_j)\tau_i$. We construct a function f in between f_1 and f_2 such that $f_2-f>0$ and $f-f_1>0$. Assume that $f(\mu)=\ell(1-c-c\mu\tau)-\sum_{i=1}^{\ell}\sum_{j=1}^{r}q_j(\tau_i-\sigma_j)$.

Then,

$$\begin{aligned} f-f_1&=\ell(1-c-c\mu\tau)-\sum_{i=1}^{\ell}\sum_{j=1}^{r}q_j(\tau_i-\sigma_j)-\sum_{i=1}^{\ell}\left(1-ce^{\mu\tau}-\sum_{j=1}^{r}q_je^{\mu\sigma_j}\int_0^{\tau_i-\sigma_j}e^{\mu s}\,ds\right)\\ &=\sum_{i=1}^{\ell}\left(1-ce^{\mu\tau}-\sum_{j=1}^{r}q_je^{\mu\sigma_j}\int_0^{\tau_i-\sigma_j}e^{\mu s}\,ds\right)\\ &=\sum_{i=1}^{\ell}\left\{c\left(e^{\mu\tau}-1-\mu\tau\right)+\sum_{j=1}^{r}q_j\left[e^{\mu\sigma_j}\int_0^{\tau_i-\sigma_j}e^{\mu s}\,ds-(\tau_i-\sigma_j)\right]\right\}>0,\quad\forall\mu>0,\\ f_2-f&=\sum_{i=1}^{\ell}g_i(\mu), \end{aligned} \tag{2.12}$$

where

$$g_i(\mu)=\left\{\ell p_i-\sum_{j=1}^{r}q_j\right\}\frac{e^{\mu\tau_i}}{\mu}-(1-c-c\mu\tau)+\sum_{j=1}^{r}q_j(\tau_i-\sigma_j),\quad i=1,2,\ldots,\ell. \tag{2.13}$$

Since $e^{\mu\tau_i}/\mu$ has a minimum value at $\mu=1/\tau_i$, then $e^{\mu\tau_i}\geq e\tau_i$, for all $\mu>0$, $i=1,2,\ldots,\ell$. Hence,

$$\begin{aligned} g_i(\mu)\big|_{\mu=1/\alpha\tau_i}&=\left\{\ell p_i-\sum_{j=1}^{r}q_j\right\}\alpha\tau_ie^{i/\alpha}-\left(1-c-c\frac{\tau}{\alpha\tau_i}\right)+\sum_{j=1}^{r}q_j(\tau_i-\sigma_j),\quad i=1,2,\ldots,\ell\\ &>\left\{\ell p_i-\sum_{j=1}^{r}q_j\right\}\alpha\tau_i-(1-c),\quad\alpha\geq1. \end{aligned} \tag{2.14}$$

For

$$\alpha > \frac{1-c}{(\ell p_i - \sum_{j=1}^{r} q_j)\tau_i}, \quad i = 1, 2, \dots, \ell, \tag{2.15}$$

we have, $g_i(\mu) > 0$ for $\mu \in (0, 1/\alpha\tau_i)$. It follows that $g_i(\mu) > 0$, for all $\mu \in (0, (\ell p_i - \sum_{j=1}^{r} q_j)\tau_i/(1-c))$. We now consider $\mu \geq (\ell p_i - \sum_{j=1}^{r} q_j)\tau_i/(1-c)$ and note that

$$[f_2 - f]_\mu \geq \sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right)\left(e + \frac{c\tau}{1-c}\right)\tau_i - \sum_{i=1}^{\ell}\left((1-c) - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right) > 0, \tag{2.16}$$

since,

$$\left(e + \frac{c\tau}{1-c}\right)\sum_{i=1}^{\ell}\left(\ell p_i - \sum_{j=1}^{r} q_j\right)\tau_i > \sum_{i=1}^{\ell}\left((1-c) - \sum_{j=1}^{r} q_j(\tau_i - \sigma_j)\right). \tag{2.17}$$

□

Example 2.2. Consider the neutral delay differential equation in the form

$$\frac{d}{dt}(x(t) - cx(t-2\pi)) + x(t-4\pi) - x(t-2\pi) + (1-c)x\left(t - \frac{3\pi}{2}\right) = 0. \tag{2.18}$$

This equation has an oscillatory solution $x(t) = \sin t$ but not all solutions are oscillatory since the sufficient conditions—in Theorem 2.1—are not satisfied. In fact, it has a nonoscillatory solution $x(t) = e^{-\lambda t}$, $0.0608314 < \lambda < 0.0608315$.

References

[1] H. A. Agwo, *On the oscillation of delay differential equations with real coefficients*, Int. J. Math. Math. Sci. **22** (1999), no. 3, 573–578.

[2] O. Arino and I. Györi, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differential Equations **81** (1989), no. 1, 98–105.

[3] R. K. Brayton and R. A. Willoughby, *On the numerical integration of a symmetric system of difference-differential equations of neutral type*, J. Math. Anal. Appl. **18** (1967), 182–189.

[4] R. D. Driver, *A mixed neutral system*, Nonlinear Anal. **8** (1984), no. 2, 155–158.

[5] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer, Dordrecht, 1992.

[6] K. Gopalsamy and B. G. Zhang, *Oscillation and nonoscillation in first order neutral differential equations*, J. Math. Anal. Appl. **151** (1990), no. 1, 42–57.

[7] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, Clarendon Press, New York, 1991.

[8] J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

[9] G. Ladas and S. W. Schults, *On oscillations of neutral equations with mixed arguments*, Hiroshima Math. J. **19** (1989), no. 2, 409–429.

[10] G. Ladas and Y. G. Sficas, *Oscillations of neutral delay differential equations*, Canad. Math. Bull. **29** (1986), no. 4, 438–445.

[11] A. I. Zahariev and D. D. Baĭnov, *Oscillating properties of the solutions of a class of neutral type functional-differential equations*, Bull. Austral. Math. Soc. **22** (1980), no. 3, 365–372.

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