IJMMS 29:10 (2002) 609–611
PII. S0161171202007974
http://ijmms.hindawi.com

A GENERALIZATION RELATED TO SCHRÖDINGER OPERATORS WITH A SINGULAR POTENTIAL

TOKA DIAGANA

Received 30 July 2001

The purpose of this note is to generalize a result related to the Schrödinger operator $L = -\Delta + Q$, where Q is a singular potential. Indeed, we show that $D(L) = \{0\}$ in $L^2(\mathbb{R}^d)$ for $d \geq 4$. This fact answers to an open question that we formulated.

2000 Mathematics Subject Classification: 35J05, 47B25, 47B44.

1. Introduction. Let L be the operator of Schrödinger, defined in $L^2(\mathbb{R}^d)$ as $L = A + B$, where

$$\begin{aligned} A\phi &= -\Delta\phi, \qquad D(A) = H^2(\mathbb{R}^d), \\ B\phi &= Q\phi, \qquad D(B) = \{\phi \in L^2(\mathbb{R}^d) : Q\phi \in L^2(\mathbb{R}^d)\}. \end{aligned} \tag{1.1}$$

We suppose that the potential Q, verifies the following conditions, see, for example, [1],

$$Q > 0, \qquad Q \in L^1(\mathbb{R}^d), \qquad Q \notin L^2_{\text{loc}}(\mathbb{R}^d). \tag{1.2}$$

Under these conditions we show that $D(L) = \{0\}$, for $d \geq 4$, this fact extends the author's result (the case where $d \leq 3$, see the details in [1]). For that we use approximations of functions of $H^2(\mathbb{R}^d)$ (when $d \geq 4$) by continuous functions in connection with BMO space (where $\text{BMO}(\mathbb{R}^d)$ is the space of functions of Bounded Mean Oscillation), see [2]. Let $\phi \in L^2(\mathbb{R}^d)$, we denote by I_α the operator defined by

$$I_\alpha\phi = (-\Delta)^{-\alpha/2}\phi = \sqrt{(-\Delta)}^{(-\alpha)}\phi. \tag{1.3}$$

Thus, we know that

$$||I_\alpha\phi||_{L^q(\mathbb{R}^d)} \leq C_{p,q,d}\|\phi\|_{L^p(\mathbb{R}^d)}, \quad \text{if } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d} \text{ with } \frac{1}{p} > \frac{\alpha}{d}. \tag{1.4}$$

In the case where $1/p = \alpha/d$, $\alpha = p = 2$, and $d = 4$, then

$$||I_2\phi||_{\text{BMO}} \leq C\|\phi\|_2. \tag{1.5}$$

We also have

$$I_2(C_0^\infty(\mathbb{R}^d)) \subseteq \text{VMO}, \tag{1.6}$$

where $\text{VMO}(\mathbb{R}^d)$ is the space of functions of Vanishing Mean Oscillation. We can find details in [2].

2. Generalization. Let H be a Hilbert space given by $H = L^2(\mathbb{R}^d)$, thus we have the following proposition.

Proposition 2.1. *Under the previous hypotheses on the singular potential Q and if $d \geq 4$, then*

$$D(L) = \{0\}. \tag{2.1}$$

Proof. Let $u \in D(A) \cap D(B)$, suppose that $u \neq 0$, then there exists an open subset Ω of $\mathbb{R}^d$ such that $|u(x)| > a$, for all $x \in \Omega \subseteq \operatorname{supp} u$. Let $\Omega' \subseteq \Omega$, be a compact subset of Ω.

Step 1. When $d \leq 3$, done in [1].

Step 2. Suppose $d = 4$. Then there exists $(u_k) \in C_0^\infty(\mathbb{R}^4)$ such that u_k converges to u into $H^2(\mathbb{R}^4)$, thus, we can write $u_k = I_2 v_k$ and $u = I_2 v$ and $v \in L^2(\mathbb{R}^4)$. It follows that

$$||u_k - u||_{\mathrm{BMO}} \leq C||v_k - v||_2 \longrightarrow 0 \tag{2.2}$$

because v_k converges to v into $L^2(\mathbb{R}^4)$, thus u_k converges to u into BMO. Consider u_k and u as functions defined on Ω', then $|Q|_{|\Omega'} = (|Qu_k|/|u|_k)_{|\Omega'}$, on passing to the limit in BMO and by the fact that B is a closed operator. It follows that $Q \in L^2(\Omega')$, that is impossible according to the hypothesis on the potential, $Q \notin L^2_{\mathrm{loc}}(\mathbb{R}^4)$. And then, we conclude that $u = 0$.

Step 3. Suppose $d > 4$, and write $u_k = I_2 V_k$ and $u = I_2 v$ where v_k converges to v into $L^2(\mathbb{R}^d)$. Thus, $\alpha = p = 2$ and $1/q = 1/2 - 2/d$ where $d > 4$, therefore,

$$||u_k - u||_q = ||I_2 v_k - I_2 v||_q \leq C||v_k - v||_2, \tag{2.3}$$

then u_k converges to u into $L^q(\mathbb{R}^d)$. We also write, $Q = Qu_k/u_k$, and consider this function on Ω' and by passing to the limit into $L^q(\mathbb{R}^d)$, we get a contradiction. □

Conclusion. The domain of the algebraic sum of A and B is always zero, that is, $D(A) \cap D(B) = \{0\}$, without restriction on d.

Remarks. The dimensional d of $\mathbb{R}^d$ has no impact on the sum form of A and B, $(-\Delta \oplus Q)$. This operator is always defined and verifies Kato's condition and is given as

$$\begin{gathered} D((-\Delta \oplus Q)) = \{u \in H^1(\mathbb{R}^d) : Q|u|^2 \in L^1(\mathbb{R}^d),\ -\Delta u + Qu \in L^2(\mathbb{R}^d)\}, \\ (-\Delta \oplus Q)u = -\Delta u + Qu \end{gathered} \tag{2.4}$$

therefore, Kato's condition is satisfied, that is,

$$D\left(\sqrt{(-\Delta \oplus Q)}\right) = D(\sqrt{-\Delta}) \cap D\left(\sqrt{Q}\right) = D\left(\sqrt{(-\Delta \oplus Q)*}\right). \tag{2.5}$$

The example of singular potential given in [1] is always valid for all d,

$$Q(x) = \sum_{k=0}^{+\infty} \frac{G(x - \alpha_k)}{k^2}, \tag{2.6}$$

where G is a function defined on the compact subset Ω of $\mathbb{R}^d$ and verifying

$$G > 0, \quad G \in L^1(\Omega), \quad G \notin L^2(\Omega), \quad G = 0 \quad \text{on } \mathbb{R}^d - \Omega, \tag{2.7}$$

where $\alpha_k = (\alpha_k^1, \alpha_k^2, \dots, \alpha_k^d) \in Q^d$ is a rational sequence.

Acknowledgment. I want to express my thanks to Professor Steve Hofmann, from University of Missouri at Columbia about the discussions we had had on the point solved in this note.

References

[1] T. Diagana, *Schrödinger operators with a singular potential*, Int. J. Math. Math. Sci. **29** (2002), no. 6, 371–373.

[2] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, no. 30, Princeton University Press, New Jersey, 1970.

Toka Diagana: Department of Mathematics, Howard University, 2441 Sixth Street, NW, Washington, DC 20059, USA

E-mail address: tdiagana@howard.edu