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RELATIONSHIPS OF CONVOLUTION PRODUCTS, GENERALIZED TRANSFORMS, AND THE FIRST VARIATION ON FUNCTION SPACES

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We use a generalized Brownian motion process to define the generalized Fourier-Feynman transform, the convolution product, and the first variation. We then examine the various relationships that exist among the first variation, the generalized Fourier-Feynman transform, and the convolution product for functionals on function space that belong to a Banach algebra $S(L_{ab}[0,T])$. These results subsume similar known results obtained by Park, Skoug, and Storvick (1998) for the standard Wiener process.

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1. Introduction. The concept of L_1 analytic Fourier-Feynman transform (FFT) was introduced by Brue in [1]. In [3], Cameron and Storvick introduced an L_2 analytic FFT. In [13], Johnson and Skoug developed an L_p analytic FFT for $1 \le p \le 2$, which extended the results in [1, 3] and gave various relationships between the L_1 and L_2 theories. In [10], Huffman et al. defined a convolution product for functionals on Wiener space and obtained, in [11, 12], various results involving the FFT and the convolution product.

Both the FFT and the convolution product are defined in terms of a Feynman integral. In this paper, we extend the ideas of [10, 11, 12] from the Wiener process to more general stochastic process. We note that the Wiener process is free of drift and is stationary in time. However, the stochastic process considered in this paper is subject to drift and nonstationary in time.

In Section 2, we consider the function space induced by a generalized Brownian motion process and define several concepts. In Section 3, we examine all relationships involving exactly two of three concepts of transform, convolution product and first variation of functionals in $S(L_{ab}[0,T])$. In Section 4, we examine all relationships involving all three of theses concepts where each concept is used exactly once.

2. Definitions and preliminaries. Let $D = [0,T]$ and let $(\Omega,\mathcal{B},P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega,\mathcal{B},P)$ and D is called *a generalized Brownian motion process* if $Y(0,\omega) = 0$ a.e.; and for $0 \le t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1,\omega),\dots,Y(t_n,\omega))$ is normally distributed with the density function

$$K(\vec{t},\vec{\eta}) = \left((2\pi)^n \prod_{j=1}^{n}\left(b(t_j)-b(t_{j-1})\right)\right)^{-1/2} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\frac{\left((\eta_j-a(t_j))-(\eta_{j-1}-a(t_{j-1}))\right)^2}{b(t_j)-b(t_{j-1})}\right\}, \tag{2.1}$$

where $\vec{\eta} = (\eta_1,\dots,\eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1,\dots,t_n)$, $t_0 = 0$, and $a(t)$ is a continuous real-valued function of bounded variation with $a(0) = 0$, and $b(t)$ is a strictly increasing, continuous real-valued function with $b(0) = 0$.

As explained in [19, pages 18–20], Y induces a probability measure μ on the measurable space $(\mathbb{R}^D,\mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps, $e_t(x) = x(t)$, defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D,\mathcal{B}^D,\mu)$ is a probability measure space.

We note that, the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s),b(t)\}$. By [19, Theorem 14.2, page 187], the probability measure μ induced by Y, taking a separable version, is supported by $C_{ab}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{ab}[0,T],\mathcal{B}(C_{ab}[0,T]),\mu)$ is the function space induced by Y where $\mathcal{B}(C_{ab}[0,T])$ is the Borel σ-algebra of $C_{ab}[0,T]$. Note that we can also express x in the form

$$x(t) = w(b(t)) + a(t), \quad 0 \le t \le T, \tag{2.2}$$

where $w(\cdot)$ is the standard Brownian motion process [6, 7].

Let $L_{ab}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$, that is,

$$L_{ab}[0,T] = \left\{v : \int_0^T v^2(s)\,db(s) < \infty \text{ and } \int_0^T v^2(s)\,d|a|(s) < \infty\right\}, \tag{2.3}$$

where $|a|$ denotes the total variation [6, 7].

A subset B of $C_{ab}[0,T]$ is said to be scale-invariant measurable [8, 14] provided that ρB is $\mathcal{B}(C_{ab}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null set provided that $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set, is said to hold on scale-invariant almost everywhere (s-a.e.). If two functionals F and G defined on $C_{ab}[0,T]$ are equal s-a.e., we write $F \approx G$.

We denote the function space integral of a $\mathcal{B}(C_{ab}[0,T])$-measurable functional F by

$$E[F] = \int_{C_{ab}[0,T]} F(x)\,d\mu(x) \tag{2.4}$$

whenever the integral exists.

We are now ready to state the definitions of the generalized analytic Feynman integral.

Definition 2.1. Let $\mathbb{C}$ denote the complex numbers. Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$ and $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$. Let $F : C_{ab}[0,T] \to \mathbb{C}$ be such that, for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{ab}[0,T]} F(\lambda^{-1/2}x)\,d\mu(x) \tag{2.5}$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{ab}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$E^{an_\lambda}[F] = J^*(\lambda). \tag{2.6}$$

Let $q \neq 0$ be a real number and let F be a functional such that $E^{an_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral of F with parameter q and we write

$$E^{anf_q}[F] = \lim_{\lambda \to -iq} E^{an_\lambda}[F], \tag{2.7}$$

where λ approaches $-iq$ through $\mathbb{C}_+$.

Next we state the definition of the generalized FFT (GFFT).

Definition 2.2. For $\lambda > 0$ and $y \in C_{ab}[0,T]$, let

$$T_\lambda(F)(y) = E_x^{an_\lambda}[F(y+x)]. \tag{2.8}$$

In the standard Fourier theory, the integrals involved are often interpreted in the mean; a similar concept is useful in the FFT theory [13, page 104]. Let $p \in (1,2]$ and let p and p' be related by $1/p + 1/p' = 1$. Let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} E\Big[\,|H_n(\rho y) - H(\rho y)|^{p'}\Big] = 0. \tag{2.9}$$

Then we write

$$H \approx \underset{n\to\infty}{\text{l.i.m.}}\, H_n \tag{2.10}$$

and we call H the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ. Let real $q \neq 0$ be given. For $1 < p \leq 2$, we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(p)}(F)(y) = \underset{\lambda\to -iq}{\text{l.i.m.}}\, T_\lambda(F)(y) \tag{2.11}$$

if it exists. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula ($\lambda \in \mathbb{C}_+$)

$$T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq} T_\lambda(F)(y) \tag{2.12}$$

if it exists.

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

Definition 2.3. Let F and G be functionals on $C_{ab}[0,T]$. For $\lambda \in \tilde{\mathbb{C}}_+$ we define the convolution product (if it exists) by

$$(F * G)_\lambda(y) = \begin{cases} \int_{C_{ab}[0,T]}^{\mathrm{an}_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) d\mu(x), & \lambda \in \mathbb{C}_+ \\ \int_{C_{ab}[0,T]}^{\mathrm{anf}_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) d\mu(x), & \lambda = -iq. \end{cases} \tag{2.13}$$

Remark 2.4. When $\lambda = -iq$, we denote $(F * G)_\lambda$ by $(F * G)_q$.

We finish this section by giving the definition of the first variation δF of the functional F [2, 5].

Definition 2.5. Let F be a $\mathcal{B}(C_{ab}[0,T])$-measurable functional on $C_{ab}[0,T]$ and let $w \in C_{ab}[0,T]$. Then

$$\delta F(x \mid w) = \frac{\partial}{\partial h} F(x + hw)\Big|_{h=0} \tag{2.14}$$

(if it exists) is called the first variation of F.

The following analytic Feynman integration formula is used throughout:

$$E^{\mathrm{anf}_q}[\exp\{i\langle v, x\rangle\}] = \exp\left\{-\frac{i}{2q}(v^2, b) + i\left(\frac{i}{q}\right)^{1/2}(v, a)\right\} \tag{2.15}$$

for all real $q \neq 0$, where $v \in L_{ab}[0,T]$, $\langle v, x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral $\int_0^T v(s)dx(s)$, and (v^2, b) denotes the Lebesgue Stieltjes integral $\int_0^T v^2(s)db(s)$.

3. Relationships involving exactly two of three concepts of transform, convolution, and first variation. Let $M(L_{ab}[0,T])$ be the space of $\mathbb{C}$-valued, countably additive finite Borel measures on $L_{ab}[0,T]$. The Banach algebra $S(L_{ab}[0,T])$ consists of functionals F on $C_{ab}[0,T]$ expressible in the form

$$F(x) = \int_{L_{ab}[0,T]} \exp\{i\langle u, x\rangle\} df(u), \tag{3.1}$$

for s-a.e. $x \in C_{ab}[0,T]$, where f is an element of $M(L_{ab}[0,T])$. Further works on $S(L_{ab}[0,T])$ show that it contains many functionals of interest in Feynman integration theory [3, 4, 6, 7, 13, 14, 15, 16, 17].

Also, let

$$A \equiv \{y \in C_{ab}[0,T] : y \text{ is absolutely continuous on } [0,T] \text{ with } y' \in L_{ab}[0,T]\}. \tag{3.2}$$

Remark 3.1. Throughout, we choose the variance function $b(\cdot)$ which is strictly increasing such that the function p defined by $p(t) = 1/b'(t)$, $p(0) = p(T) = 0$ is of bounded variation on $[0,T]$. For any $u \in L_{ab}[0,T]$, let

$$\|u\|_b = \left(\int_0^T u^2(s)db(s)\right)^{1/2} = \sqrt{(u^2, b)}, \tag{3.3}$$

then $\|\cdot\|_b$ is a norm on $L_{ab}[0,T]$ since $\int_0^T u^2(s)db(s) < \infty$. Hence we have for any $u,v \in L_{ab}[0,T]$,

$$|(uv,b)| \le \|u\|_b \|v\|_b. \tag{3.4}$$

This will insure that the first variation, $\delta F(\cdot \mid w)$ of F in $S(L_{ab}[0,T])$, that arises will exist for all $w \in A$ (see [9]).

Let G in $S(L_{ab}[0,T])$ be given by

$$G(x) = \int_{L_{ab}[0,T]} \exp\{i\langle v,x\rangle\}dg(v) \tag{3.5}$$

for s-a.e. $x \in C_{ab}[0,T]$ where $g \in M(L_{ab}[0,T])$.

In our first lemma, we obtain a formula for the first variation of functionals in $S(L_{ab}[0,T])$.

Lemma 3.2. *Let $F \in S(L_{ab}[0,T])$ be given by (3.1) with $\int_{L_{ab}[0,T]} \|u\|_b |df(u)| < \infty$. Then for each $w \in A$ and s-a.e. $y \in C_{ab}[0,T]$,*

$$\delta F(y \mid w) = \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\{i\langle u,y\rangle\}df(u). \tag{3.6}$$

Furthermore, as a function of y, $\delta F(y \mid w)$ is an element of $S(L_{ab}[0,T])$.

Proof. By using the definition of the first variation, we see that

$$\begin{aligned} \delta F(y \mid w) &= \frac{\partial}{\partial h}\left(\int_{L_{ab}[0,T]} \exp\{i\langle u,y\rangle + ih\langle u,w\rangle\}df(u)\right)\Bigg|_{h=0} \\ &= \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\{\langle u,y\rangle\}df(u) \end{aligned} \tag{3.7}$$

for s-a.e. $y \in C_{ab}[0,T]$. Next, let $\phi_w(E) = \int_E i\langle u,w\rangle df(u)$ for each set $E \in \mathcal{B}(L_{ab}[0,T])$. But

$$\|\phi_w\| \le \int_{L_{ab}[0,T]} |i\langle u,w\rangle| \, |df(u)| \le M\|w'\|_b \int_{L_{ab}[0,T]} \|u\|_b |df(u)| < \infty, \tag{3.8}$$

where $M = \sup_{t\in[0,T]} p(t)$. Hence $\delta F(y \mid w) = \int_{L_{ab}[0,T]} \exp\{i\langle u,y\rangle\}d\phi_w(u)$ is an element of $S(L_{ab}[0,T])$. □

In our next theorem, we obtain the transform of functional in $S(L_{ab}[0,T])$.

Theorem 3.3. *Let $F \in S(L_{ab}[0,T])$ be given by (3.1) and let $p \in [1,2]$ be given. Then the analytic generalized Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all real $q \ne 0$ and is given by the formula*

$$T_q^{(p)}(F)(y) = \int_{L_{ab}[0,T]} \exp\left\{i\langle u,y\rangle - \frac{i}{2q}(u^2,b) + i\left(\frac{i}{q}\right)^{1/2}(u,a)\right\}df(u) \tag{3.9}$$

for s-a.e. $y \in C_{ab}[0,T]$.

Proof. By (2.8), the Fubini theorem, and (2.15), we have, for all $\lambda > 0$,

$$T_\lambda(F)(y) = \int_{C_{ab}[0,T]}^{\text{an}_\lambda} F(y+x)\,d\mu(x) = \int_{L_{ab}[0,T]} \int_{C_{ab}[0,T]}^{\text{an}_\lambda} \exp\{i\langle u,y\rangle + i\langle u,x\rangle\}\,d\mu(x)\,df(u) = \int_{L_{ab}[0,T]} \exp\left\{i\langle u,y\rangle - \frac{1}{2\lambda}(u^2,b) + \frac{i}{\sqrt{\lambda}}(u,a)\right\}df(u) \tag{3.10}$$

for s-a.e. $y \in C_{ab}[0,T]$. But the last equation above is analytic throughout $\mathbb{C}_+$ and is continuous on $\tilde{\mathbb{C}}_+$, since f is a finite Borel measure. Thus (3.9) is established. □

In the following theorem, we obtain the convolution product of functionals in $S(L_{ab}[0,T])$.

Theorem 3.4. *Let $F \in S(L_{ab}[0,T])$ be given by (3.1), and let $G \in S(L_{ab}[0,T])$ be given by (3.5). Then their convolution product $(F*G)_q$ exists for all real $q \neq 0$ and is given by the formula*

$$(F*G)_q(y) = \int_{L^2_{ab}[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b) + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\}df(u)\,dg(v) \tag{3.11}$$

for s-a.e. $y \in C_{ab}[0,T]$.

Proof. By using (2.13), the Fubini theorem, and (2.15), we have that for all $\lambda > 0$,

$$(F*G)_\lambda(y) = \int_{C_{ab}[0,T]}^{\text{an}_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right)G\left(\frac{y-x}{\sqrt{2}}\right)d\mu(x) = \int_{L^2_{ab}[0,T]} \int_{C_{ab}[0,T]}^{\text{an}_\lambda} \exp\left\{\frac{i}{\sqrt{2}}\langle u-v,x\rangle + \frac{i}{\sqrt{2}}\langle u+v,y\rangle\right\}d\mu(x)\,df(u)\,dg(v) = \int_{L^2_{ab}[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{1}{4\lambda}((u-v)^2,b) + \frac{i}{\sqrt{2\lambda}}(u-v,a)\right\}df(u)\,dg(v) \tag{3.12}$$

for s-a.e. $y \in C_{ab}[0,T]$. But the last equation above is analytic throughout $\mathbb{C}_+$, and is continuous on $\tilde{\mathbb{C}}_+$. Thus we have the desired result. □

Next, we obtain the transform of the convolution product.

Theorem 3.5. *Let F and G be given as in Theorem 3.4 and $p \in [1,2]$. Then for all real $q \neq 0$, $T_q^{(p)}((F*G)_q)$ exists and is given by the formula*

$$T_q^{(p)}((F*G)_q)(y) = T_q^{(p)}(F_1)\left(\frac{y}{\sqrt{2}}\right)T_q^{(p)}(G_1)\left(\frac{y}{\sqrt{2}}\right) \tag{3.13}$$

for s-a.e. $y \in C_{ab}[0,T]$, where F_1 and G_1 are given by (3.15) below.

Proof. Let $p \in [1,2]$ and $q \in \mathbb{R}-\{0\}$. Using (2.15), (3.9), and (3.11), we see that

$$\begin{aligned}
&T_q^{(p)}((F*G)_q)(y) \\
&\quad = \int_{C_{ab}[0,T]}^{anf_q} (F*G)_q(y+z)\,d\mu(z) \\
&\quad = \int_{C_{ab}^2[0,T]}^{anf_q} F\Big(\frac{(y+z)+x}{\sqrt{2}}\Big) G\Big(\frac{(y+z)-x}{\sqrt{2}}\Big)\,d\mu(x)\,d\mu(z) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{-\frac{i}{4q}((u-v)^2,b)+i\Big(\frac{i}{2q}\Big)^{1/2}(u-v,a)\Big\} \\
&\qquad \cdot \int_{C_{ab}[0,T]}^{anf_q} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,z\rangle+\frac{i}{\sqrt{2}}\langle u+v,y\rangle\Big\}\,d\mu(z)\,df(u)\,dg(v) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{-\frac{i}{4q}((u-v)^2,b)+i\Big(\frac{i}{2q}\Big)^{1/2}(u-v,a)-\frac{i}{4q}((u+v)^2,b) \\
&\qquad\qquad +i\Big(\frac{i}{2q}\Big)^{1/2}(u+v,a)+\frac{i}{\sqrt{2}}\langle u+v,y\rangle\Big\}\,df(u)\,dg(v) \\
&\quad = \int_{L_{ab}[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u,y\rangle-\frac{i}{2q}(u^2,b)+i\Big(\frac{i}{q}\Big)^{1/2}(u,a)\Big\}\,df_1(u) \\
&\qquad \cdot \int_{L_{ab}[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle v,y\rangle-\frac{i}{2q}(v^2,b)+i\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big\}\,dg_1(v) \\
&\quad = T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\Big)
\end{aligned} \tag{3.14}$$

for s-a.e. $y \in C_{ab}[0,T]$, where

$$\begin{aligned}
F_1(y) &= \int_{L_{ab}[0,T]} \exp\{i\langle u,y\rangle\}\,df_1(u), \\
G_1(y) &= \int_{L_{ab}[0,T]} \exp\{i\langle v,y\rangle\}\,dg_1(v),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
f_1(E) &= \int_E \exp\Big\{-i\Big(\frac{i}{q}\Big)^{1/2}(u,a)+2i\Big(\frac{i}{2q}\Big)^{1/2}(u,a)\Big\}\,df(u), \\
g_1(E) &= \int_E \exp\Big\{-i\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big\}\,dg(v),
\end{aligned} \tag{3.16}$$

for every $E \in \mathcal{B}(L_{ab}[0,T])$, and so $\|f_1\| \le \|f\|$ and $\|g_1\| \le \|g\|$. □

In our next theorem, we have the convolution product of transforms of F and G in $S(L_{ab}[0,T])$.

Theorem 3.6. *Let F, G, and p be given as in Theorem 3.5. Then for all real $q \neq 0$, $(T_q^{(p)}(F) * T_q^{(p)}(G))_{-q}(y)$ exists and is given by the formula*

$$\big(T_q^{(p)}(F) * T_q^{(p)}(G)\big)_{-q}(y) = T_q^{(p)}\Big(F_2\Big(\frac{\cdot}{\sqrt{2}}\Big)G_2\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y) \tag{3.17}$$

for s-a.e. $y \in C_{ab}[0,T]$ where F_2 and G_2 are given by (3.19) below.

Proof. By using (3.9), (3.11), and the Fubini theorem, we have, for s-a.e. $y \in C_{ab}[0,T]$,

$$\begin{aligned}
&\Big(T_q^{(p)}(F) * T_q^{(p)}(G)\Big)_{-q}(y) \\
&\quad = \int_{C_{ab}[0,T]}^{anf_{-q}} T_q^{(p)}(F)\Big(\frac{y+x}{\sqrt{2}}\Big) T_q^{(p)}(G)\Big(\frac{y-x}{\sqrt{2}}\Big)\,d\mu(x) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{2q}(u^2+v^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(u+v,a)\Big\} \\
&\qquad\quad \cdot\Big[\int_{C_{ab}[0,T]}^{anf_{-q}} \exp\Big\{\frac{i}{\sqrt{2}}\langle u-v,x\rangle\Big\}\,d\mu(x)\Big]\,df(u)\,dg(v) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{2q}(u^2+v^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(u+v,a) \\
&\qquad\qquad\qquad + \frac{i}{4q}((u-v)^2,b) + \Big(-\frac{i}{2q}\Big)(u-v,a)\Big\}\,df(u)\,dg(v) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u+v)^2,b) + i\Big(\frac{i}{2q}\Big)^{1/2}(u+v,a) \\
&\qquad\qquad\qquad - i\Big(\frac{i}{2q}\Big)^{1/2}(u+v,a) + i\Big(-\frac{i}{2q}\Big)^{1/2}(u-v,a) \\
&\qquad\qquad\qquad + i\Big(\frac{i}{q}\Big)^{1/2}(u+v,a)\Big\}\,df(u)\,dg(v) \\
&\quad = \int_{L_{ab}^2[0,T]} \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u+v)^2,b) \\
&\qquad\qquad\qquad + i\Big(\frac{i}{2q}\Big)^{1/2}(u+v,a)\Big\}\,df_2(u)\,dg_2(v) \\
&\quad = T_q^{(p)}\Big(F_2\Big(\frac{\cdot}{\sqrt{2}}\Big)G_2\Big(\frac{\cdot}{\sqrt{2}}\Big)\Big)(y),
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
F_2(y) &= \int_{L_{ab}[0,T]} \exp\{i\langle u,y\rangle\}\,df_2(u), \\
G_2(y) &= \int_{L_{ab}[0,T]} \exp\{i\langle v,y\rangle\}\,dg_2(v),
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
f_2(E) &= \int_E \exp\Big\{i\Big(-\Big(\frac{i}{2q}\Big)^{1/2} + \Big(-\frac{i}{2q}\Big)^{1/2} + \Big(\frac{i}{q}\Big)^{1/2}\Big)(u,a)\Big\}\,df(u), \\
g_2(E) &= \int_E \exp\Big\{i\Big(-\Big(\frac{i}{2q}\Big)^{1/2} - \Big(-\frac{i}{2q}\Big)^{1/2} + \Big(\frac{i}{q}\Big)^{1/2}\Big)(v,a)\Big\}\,dg(v),
\end{aligned} \tag{3.20}$$

for every $E \in \mathcal{B}(L_{ab}[0,T])$, and so $\|f_2\| \le \|f\|$ and $\|g_2\| \le \|g\|$. □

In the next theorem, we obtain that the transform with respect to the first argument of the variation equals the variation of the transform.

Theorem 3.7. *Let F be given as in Lemma 3.2, $p \in [1,2]$, $q \in \mathbb{R}-\{0\}$, and $w \in A$ be given. Then*

$$T_q^{(p)}(\delta F(\cdot \mid w))(y) = \delta T_q^{(p)}(F)(y \mid w) \tag{3.21}$$

for s-a.e. $y \in C_{ab}[0,T]$.

Also, both expressions in (3.21) are given by the expression

$$\int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\left\{i\langle u,y\rangle - \frac{i}{2q}(u^2,b) + i\left(\frac{i}{q}\right)^{1/2}(u,a)\right\} df(u). \tag{3.22}$$

Proof. By using (3.6), the Fubini theorem, (2.15), and (3.9), we have that

$$\begin{aligned}
T_q^{(p)}&(\delta F(\cdot \mid w))(y) \\
&= \int_{C_{ab}[0,T]}^{anf_q} \delta F(y+x \mid w)\,d\mu(x) \\
&= \int_{C_{ab}[0,T]}^{anf_q} \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\{i\langle u,y+x\rangle\}\,df(u)\,d\mu(x) \\
&= \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\{i\langle u,y\rangle\}\left[\int_{C_{ab}[0,T]}^{anf_q} \exp\{i\langle u,x\rangle\}\,d\mu(x)\right] df(u) \\
&= \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\left\{i\langle u,y\rangle - \frac{i}{2q}(u^2,b) + i\left(\frac{i}{q}\right)^{1/2}(u,a)\right\} df(u) \\
&= \int_{L_{ab}[0,T]} \frac{\partial}{\partial h}\left(\exp\left\{i\langle u,y+hw\rangle - \frac{i}{2q}(u^2,b) + i\left(\frac{i}{q}\right)^{1/2}(u,a)\right\}\right)\Big|_{h=0} df(u) \\
&= \frac{\partial}{\partial h}\left(T_q^{(p)}(F)(y+hw)\right)\Big|_{h=0} \\
&= \delta T_q^{(p)}(F)(y \mid w)
\end{aligned} \tag{3.23}$$

for s-a.e. $y \in C_{ab}[0,T]$ as desired. □

In the next theorem, we obtain the transform with respect to the second argument of the variation.

Theorem 3.8. *Let F, p, q, and w be given as in Theorem 3.7. Then, for s-a.e. $y \in C_{ab}[0,T]$,*

$$T_q^{(p)}(\delta F(y \mid \cdot))(w) = \delta F(y \mid w) + i\left(\frac{i}{q}\right)^{1/2}\int_{L_{ab}[0,T]} (u,a)\exp\{i\langle u,y\rangle\}\,df(u). \tag{3.24}$$

Proof. Using (2.11) and (3.6), we obtain

$$\begin{aligned}
T_q^{(p)}(\delta F(y \mid \cdot))(w) &= \int_{C_{ab}[0,T]}^{anf_q} \delta F(y \mid w+x)\,d\mu(x) \\
&= \int_{C_{ab}[0,T]}^{anf_q} \int_{L_{ab}[0,T]} i\langle u,w+x\rangle \exp\{i\langle u,y\rangle\}\,df(u)\,d\mu(x)
\end{aligned}$$

$$= i\int_{L_{ab}[0,T]} \exp\{i\langle u,y\rangle\}\left[\int_{C_{ab}[0,T]}^{\text{anf}_q} \langle u, w+x\rangle d\mu(x)\right] df(u)$$

$$= i\int_{L_{ab}[0,T]} \left(\langle u,w\rangle + \left(\frac{i}{q}\right)^{1/2}(u,a)\right)\exp\{i\langle u,y\rangle\} df(u)$$

$$= \delta F(y \mid w) + i\left(\frac{i}{q}\right)^{1/2}\int_{L_{ab}[0,T]} (u,a)\exp\{i\langle u,y\rangle\} df(u) \tag{3.25}$$

for s-a.e. $y \in C_{ab}[0,T]$. In particular, if $a \in A$ then $(u,a) = \langle u,a\rangle$ and so

$$T_q^{(p)}(\delta F(y \mid \cdot))(w) = \delta F(y \mid w) + \left(\frac{i}{q}\right)^{1/2}\delta F(y \mid a). \tag{3.26}$$

□

In our next theorem, we obtain the first variation of convolution product of functionals F and G in $S(L_{ab}[0,T])$.

Theorem 3.9. *Let F, p, q, and w be given as in Theorem 3.7 and let G be given by (3.5) with $\int_{L_{ab}[0,T]} \|v\|_b |dg(v)| < \infty$. Then for s-a.e. $y \in C_{ab}[0,T]$, we obtain the formula*

$$\begin{aligned}\delta(F*G)_q(y \mid w) &= \int_{L^2_{ab}[0,T]} \frac{i}{\sqrt{2}}\langle u+v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b)\right.\\ &\qquad \left. + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v)\end{aligned} \tag{3.27}$$

for s-a.e. $y \in C_{ab}[0,T]$.

Proof. The proof of (3.27) can be obtained by using (3.6) and (3.11). □

In our next theorem, we obtain the convolution product of the first variation with respect to the first argument.

Theorem 3.10. *Let F, G, p, q, and w be given as in Theorem 3.9. Then for s-a.e. $y \in C_{ab}[0,T]$, $(\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q(y)$ exists and is given by the formula*

$$\begin{aligned}(\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q(y) &= -\int_{L^2_{ab}[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b)\right.\\ &\qquad \left. + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v).\end{aligned} \tag{3.28}$$

Proof. By using (2.13), (3.6), the Fubini theorem, and (2.15), we have

$$\begin{aligned}(\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q(y) &= \int_{C_{ab}[0,T]}^{\text{anf}_q} \delta F\left(\frac{y+x}{\sqrt{2}} \,\middle|\, w\right)\delta G\left(\frac{y-x}{\sqrt{2}} \,\middle|\, w\right) d\mu(x)\end{aligned}$$

$$= \int_{C_{ab}[0,T]}^{\text{anf}_q} \int_{L_{ab}[0,T]} i\langle u,w\rangle \exp\left\{i\left\langle u, \frac{y+x}{\sqrt{2}}\right\rangle\right\} df(u) \cdot \int_{L_{ab}[0,T]} i\langle v,w\rangle \exp\left\{i\left\langle v, \frac{y-x}{\sqrt{2}}\right\rangle\right\} dg(v)\,d\mu(x)$$

$$= -\int_{L^2_{ab}[0,T]} \int_{C_{ab}[0,T]}^{\text{anf}_q} \langle u,w\rangle\langle v,w\rangle \cdot \exp\left\{\frac{i}{\sqrt{2}}\langle u-v,x\rangle + \frac{i}{\sqrt{2}}\langle u+v,y\rangle\right\} d\mu(x)\,df(u)\,dg(v)$$

$$= -\int_{L^2_{ab}[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b) + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v) \tag{3.29}$$

for s-a.e. $y \in C_{ab}[0,T]$. □

In our next theorem, we obtain the convolution product of the first variation with respect to the second argument.

Theorem 3.11. *Let F, G, p, q, and w be given as in Theorem 3.9. Then*

$$\begin{aligned}(\delta F(y\mid\cdot) * \delta G(y\mid\cdot))_q(w) &= \frac{i}{2q}\int_{L^2_{ab}[0,T]} (uv,b)\exp\{i\langle u+v,y\rangle\}\,df(u)\,dg(v)\\ &+ \left(\delta F\left(y \,\Big|\, \frac{w}{\sqrt{2}}\right) + i\left(\frac{i}{q}\right)^{1/2}\int_{L_{ab}[0,T]}\left(u,\frac{a}{\sqrt{2}}\right)\exp\{i\langle u,y\rangle\}\,df(u)\right)\\ &\cdot\left(\delta G\left(y \,\Big|\, \frac{w}{\sqrt{2}}\right) - i\left(\frac{i}{q}\right)^{1/2}\int_{L_{ab}[0,T]}\left(v,\frac{a}{\sqrt{2}}\right)\exp\{i\langle v,y\rangle\}\,dg(v)\right)\end{aligned} \tag{3.30}$$

for s-a.e. $y \in C_{ab}[0,T]$.

Proof. For each $u, v \in L_{ab}[0,T]$, we have

$$\int_{C_{ab}[0,T]}^{\text{anf}_q} \langle u,x\rangle\langle v,x\rangle\,d\mu(x) = \frac{i}{q}(u,a)(v,a) + \frac{i}{q}(uv,b). \tag{3.31}$$

But, by using (2.13), (3.6), the Fubini theorem, and (3.31), we have

$$\begin{aligned}(\delta F(y\mid\cdot) * \delta G(y\mid\cdot))_q(w) &= \int_{C_{ab}[0,T]}^{\text{anf}_q} \delta F\left(y \,\Big|\, \frac{w+x}{\sqrt{2}}\right)\delta G\left(y \,\Big|\, \frac{w-x}{\sqrt{2}}\right) d\mu(x)\\ &= -\frac{1}{2}\int_{L^2_{ab}[0,T]}\int_{C_{ab}[0,T]}^{\text{anf}_q} \langle u,w+x\rangle\langle v,w-x\rangle \exp\{i\langle u+v,y\rangle\}\,d\mu(x)\,df(u)\,dg(v)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{L^2_{ab}[0,T]}\Big[\Big(\Big(\frac{i}{q}\Big)^{1/2}(u,a)+\langle u,w\rangle\Big)\Big(\langle v,w\rangle-\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big)-\frac{i}{q}(uv,b)\Big] \\
&\qquad\qquad \cdot \exp\{i\langle u+v,y\rangle\}df(u)dg(v) \\
&= \int_{L_{ab}[0,T]}\frac{i}{\sqrt{2}}\Big(\langle u,w\rangle+\Big(\frac{i}{q}\Big)^{1/2}(u,a)\Big)\exp\{i\langle u,y\rangle\}df(u) \\
&\quad \cdot \int_{L_{ab}[0,T]}\frac{i}{\sqrt{2}}\Big(\langle v,w\rangle-\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big)\exp\{i\langle v,y\rangle\}dg(v) \\
&\quad + \frac{i}{2q}\int_{L^2_{ab}[0,T]}(uv,b)\exp\{i\langle u+v,y\rangle\}df(u)dg(v) \\
&= \Big(\delta F\Big(y\ \Big|\ \frac{w}{\sqrt{2}}\Big)+i\Big(\frac{i}{q}\Big)^{1/2}\int_{L_{ab}[0,T]}\Big(u,\frac{a}{\sqrt{2}}\Big)\exp\{\langle u,y\rangle\}df(u)\Big) \\
&\quad \cdot \Big(\delta G\Big(y\ \Big|\ \frac{w}{\sqrt{2}}\Big)-i\Big(\frac{i}{q}\Big)^{1/2}\int_{L_{ab}[0,T]}\Big(v,\frac{a}{\sqrt{2}}\Big)\exp\{\langle v,y\rangle\}dg(v)\Big) \\
&\quad + \frac{i}{2q}\int_{L^2_{ab}[0,T]}(uv,b)\exp\{i\langle u+v,y\rangle\}df(u)dg(v)
\end{aligned} \tag{3.32}
$$

for s-a.e. $y \in C_{ab}[0,T]$. In particular, if $a \in A$ then $(u,a/\sqrt{2}) = \langle u,a/\sqrt{2}\rangle$ and $(v,a/\sqrt{2}) = \langle v,a/\sqrt{2}\rangle$ and so we have

$$
\begin{aligned}
&(\delta F(y\mid\cdot)*\delta G(y\mid\cdot))_q(w) \\
&\quad = \Big(\delta F\Big(y\ \Big|\ \frac{w}{\sqrt{2}}\Big)+i\Big(\frac{i}{q}\Big)^{1/2}\delta F\Big(y,\frac{a}{\sqrt{2}}\Big)\Big)\Big(\delta G\Big(y\ \Big|\ \frac{w}{\sqrt{2}}\Big)-i\Big(\frac{i}{q}\Big)^{1/2}\delta G\Big(y,\frac{a}{\sqrt{2}}\Big)\Big) \\
&\qquad + \frac{i}{2q}\int_{L^2_{ab}[0,T]}(uv,b)\exp\{i\langle u+v,y\rangle\}df(u)dg(v).
\end{aligned} \tag{3.33}
$$

Also, by using (3.24), the alternative expression in (3.30) is given by

$$
\begin{aligned}
&-T_q^{(p)}\Big(\delta F\Big(y\ \Big|\ \frac{\cdot}{\sqrt{2}}\Big)\Big)(w)T_q^{(p)}\Big(\delta G\Big(y\ \Big|\ \frac{\cdot}{\sqrt{2}}\Big)\Big)(-w) \\
&\qquad + \frac{i}{2q}\int_{L^2_{ab}[0,T]}(uv,b)\exp\{i\langle u+v,y\rangle\}df(u)dg(v).
\end{aligned} \tag{3.34}
$$

Thus we have the desired result. □

4. Relationships involving three concepts. In this section, we look at all the relationships involving the transform, the convolution, and the first variation where each operation is used exactly once.

In our next theorem, we obtain the formula for transform with respect to the first argument of the variation of the convolution product which equals the variation of the transform of the convolution product.

Theorem 4.1. *Let F, G, p, q, and w be given as in Theorem 3.9. Then for s-a.e. $y \in C_{ab}[0,T]$, $T_q^{(p)}(\delta(F*G)_q(\cdot \mid w))(y)$ exists and is given by the formula*

$$T_q^{(p)}(\delta(F*G)_q(\cdot \mid w))(y) = T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}\Big(\delta G_1\Big(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\Big)\Big)\Big(\frac{y}{\sqrt{2}}\Big) + T_q^{(p)}\Big(\delta F_1\Big(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\Big)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\Big), \tag{4.1}$$

where F_1 and G_1 are given by (3.15).

Proof. By using (3.21) we have

$$T_q^{(p)}(\delta(F*G)_q(\cdot \mid w))(y) = \delta T_q^{(p)}((F*G)_q)(y \mid w). \tag{4.2}$$

Also, using (3.6) and (3.13), we obtain

$$\begin{aligned}
&\delta T_q^{(p)}((F*G)_q)(y \mid w)\\
&\quad= \frac{\partial}{\partial h}\Big(T_q^{(p)}(F_1)\Big(\frac{y+hw}{\sqrt{2}}\Big)T_q^{(p)}(G_1)\Big(\frac{y+hw}{\sqrt{2}}\Big)\Big)\Big|_{h=0}\\
&\quad= \frac{\partial}{\partial h}\Big[\int_{L_{ab}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}\langle u, y+hw\rangle - \frac{i}{2q}(u^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(u,a)\Big\}df_1(u)\\
&\qquad\cdot\int_{L_{ab}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}\langle v, y+hw\rangle - \frac{i}{2q}(v^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big\}dg_1(v)\Big]\Big|_{h=0}\\
&\quad= \int_{L_{ab}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}\langle u, y\rangle - \frac{i}{2q}(u^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(u,a)\Big\}df_1(u)\\
&\qquad\cdot\int_{L_{ab}[0,T]}\frac{i}{\sqrt{2}}\langle v,w\rangle\exp\Big\{\frac{i}{\sqrt{2}}\langle v, y\rangle - \frac{i}{2q}(v^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big\}dg_1(v)\\
&\qquad+\int_{L_{ab}[0,T]}\frac{i}{\sqrt{2}}\langle u,w\rangle\exp\Big\{\frac{i}{\sqrt{2}}\langle u, y\rangle - \frac{i}{2q}(u^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(u,a)\Big\}df_1(u)\\
&\qquad\cdot\int_{L_{ab}[0,T]}\exp\Big\{\frac{i}{\sqrt{2}}\langle v, y\rangle - \frac{i}{2q}(v^2,b) + i\Big(\frac{i}{q}\Big)^{1/2}(v,a)\Big\}dg_1(v)\\
&\quad= T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\Big)\delta T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\,\Big|\,\frac{w}{\sqrt{2}}\Big) + \delta T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\,\Big|\,\frac{w}{\sqrt{2}}\Big)T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\Big)\\
&\quad= T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}\Big(\delta G_1\Big(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\Big)\Big)\Big(\frac{y}{\sqrt{2}}\Big)\\
&\qquad+ T_q^{(p)}\Big(\delta F_1\Big(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\Big)\Big)\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\Big)
\end{aligned} \tag{4.3}$$

for s-a.e. $y \in C_{ab}[0,T]$. □

In our next theorem, we obtain the transform with respect to the second argument of the variation of the convolution product.

Theorem 4.2. *Let F, G, p, q, and w be given as in Theorem 4.1. Then for s-a.e. $y \in C_{ab}[0,T]$,*

$$\begin{aligned}
&T_q^{(p)}(\delta(F*G)_q(y\mid\cdot))(w)\\
&\quad= \frac{i}{\sqrt{2}}\int_{L^2_{ab}[0,T]}\Big[\langle u+v,w\rangle + \Big(\frac{i}{q}\Big)^{1/2}(u+v,a)\Big]\\
&\qquad\qquad\cdot \exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b) + i\Big(\frac{i}{2q}\Big)^{1/2}(u-v,a)\Big\}df(u)dg(v)\\
&\quad= \delta(F*G)_q(y\mid w) + \frac{i}{\sqrt{2}}\Big(\frac{i}{q}\Big)^{1/2}\\
&\qquad\cdot\int_{L^2_{ab}[0,T]}(u+v,a)\exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b)\\
&\qquad\qquad\qquad + i\Big(\frac{i}{2q}\Big)^{1/2}(u-v,a)\Big\}df(u)dg(v).
\end{aligned} \tag{4.4}$$

Proof. By using (3.11), (3.24), and (3.27), we obtain (4.4) above. In particular, if $a \in A$, then $(u+v,a) = \langle u+v,a\rangle$ and hence we have

$$T_q^{(p)}(\delta(F*G)_q(y\mid\cdot))(w) = \delta(F*G)_q(y\mid w) + \Big(\frac{i}{q}\Big)^{1/2}\delta(F*G)_q(y\mid a). \tag{4.5}$$

□

Now we obtain formulas for the transforms of the convolution product with respect to the first argument of the variations.

Theorem 4.3. *Let F, G, p, q, and w be given as in Theorem 4.1. Then, for s-a.e. $y \in C_{ab}[0,T]$,*

$$\begin{aligned}
&T_q^{(p)}((\delta F(\cdot\mid w) * \delta G(\cdot\mid w))_q)(y)\\
&\quad= T_q^{(p)}(\delta F_1(\cdot\mid w))\Big(\frac{y}{\sqrt{2}}\Big)T_q^{(p)}(\delta G_1(\cdot\mid w))\Big(\frac{y}{\sqrt{2}}\Big)\\
&\quad= \delta T_q^{(p)}(F_1)\Big(\frac{y}{\sqrt{2}}\Big| w\Big)\delta T_q^{(p)}(G_1)\Big(\frac{y}{\sqrt{2}}\Big| w\Big),
\end{aligned} \tag{4.6}$$

where F_1 and G_1 are given by (3.15) and

$$\begin{aligned}
&\int_{C_{ab}[0,T]}^{\mathrm{anf}_q}(\delta F(\cdot\mid w+x) * \delta G(\cdot\mid w+x))_q(y)\,d\mu(x)\\
&\quad= \int_{C_{ab}[0,T]}^{\mathrm{anf}_q} T_q^{(p)}\Big(\delta F\Big(\frac{y+x}{\sqrt{2}}\Big|\cdot\Big)\Big)(w)T_q^{(p)}\Big(\delta G\Big(\frac{y-x}{\sqrt{2}}\Big|\cdot\Big)\Big)(w)\,d\mu(x)\\
&\qquad - \frac{i}{q}\int_{L^2_{ab}[0,T]}(uv,b)\exp\Big\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{4q}((u-v)^2,b)\\
&\qquad\qquad\qquad + i\Big(\frac{i}{2q}\Big)^{1/2}(u-v,a)\Big\}df(u)dg(v).
\end{aligned} \tag{4.7}$$

Proof. By using (3.13) and (3.21), we obtain (4.6) above. To establish (4.7), we note that, by the use of (2.15), (3.24), (3.28), and (3.31),

$$\begin{aligned}
&\int_{C_{ab}[0,T]}^{\mathrm{anf}_q} (\delta F(\cdot \mid w+x) * \delta G(\cdot \mid w+x))_q(y)\,d\mu(x) \\
&= -\int_{C_{ab}[0,T]}^{\mathrm{anf}_q} \int_{L^2_{ab}[0,T]} \langle u, w+x\rangle \langle v, w+x\rangle \\
&\quad \cdot \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}((u-v)^2, b) \right. \\
&\qquad \left. + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v)\,d\mu(x) \\
&= -\int_{L^2_{ab}[0,T]} \left[\left(\langle u,w\rangle + \left(\frac{i}{q}\right)^{1/2}(u,a)\right)\left(\langle v,w\rangle + \left(\frac{i}{q}\right)^{1/2}(v,a)\right) + \frac{i}{q}(uv,b)\right] \\
&\quad \cdot \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}((u-v)^2, b) + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v) \\
&= \int_{C_{ab}[0,T]}^{\mathrm{anf}_q} \int_{L_{ab}[0,T]} i\left(\langle u,w\rangle + \left(\frac{i}{q}\right)^{1/2}(u,a)\right) \exp\left\{ i\left\langle u, \frac{y+x}{\sqrt{2}}\right\rangle\right\} df(u) \\
&\quad \cdot \int_{L_{ab}[0,T]} i\left(\langle v,w\rangle + \left(\frac{i}{q}\right)^{1/2}(v,a)\right) \exp\left\{ i\left\langle v, \frac{y-x}{\sqrt{2}}\right\rangle\right\} dg(v)\,d\mu(x) \\
&\quad - \frac{i}{q}\int_{L^2_{ab}[0,T]} (uv,b) \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}((u-v)^2, b) \right. \\
&\qquad \left. + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v) \\
&= \int_{C_{ab}[0,T]}^{\mathrm{anf}_q} T_q^{(p)}\left(\delta F\left(\frac{y+x}{\sqrt{2}} \,\middle|\, \cdot\right)\right)(w)\, T_q^{(p)}\left(\delta G\left(\frac{y-x}{\sqrt{2}} \,\middle|\, \cdot\right)\right)(w)\,d\mu(x) \\
&\quad - \frac{i}{q}\int_{L^2_{ab}[0,T]} (uv,b) \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i}{4q}((u-v)^2, b) \right. \\
&\qquad \left. + i\left(\frac{i}{2q}\right)^{1/2}(u-v,a)\right\} df(u)\,dg(v)
\end{aligned} \tag{4.8}$$

for s-a.e. $y \in C_{ab}[0,T]$. □

In our next theorem, we obtain the transforms of the convolution product with respect to the second argument of the variations.

Theorem 4.4. *Let F, G, p, q, and w be given as in Theorem 4.1. Then for s-a.e. $y \in C_{ab}[0,T]$,*

$$\begin{aligned}
&T_q^{(p)}((\delta F(y \mid \cdot) * \delta G(y \mid \cdot))_q)(w) \\
&\quad = T_q^{(p)}\left(\delta F\left(y \,\middle|\, \frac{\cdot}{\sqrt{2}}\right)\delta G\left(y \,\middle|\, \frac{\cdot}{\sqrt{2}}\right)\right)(w)
\end{aligned}$$

$$-i\left(\frac{i}{q}\right)^{1/2} T_q^{(p)}\left(\delta F\left(y \,\Big|\, \frac{\cdot}{\sqrt{2}}\right)\right)(w)\int_{L_{ab}[0,T]}\left(v,\frac{a}{\sqrt{2}}\right)\exp\{i\langle v,y\rangle\}dg(v)$$
$$+i\left(\frac{i}{q}\right)^{1/2} T_q^{(p)}\left(\delta G\left(y \,\Big|\, \frac{\cdot}{\sqrt{2}}\right)\right)(w)\int_{L_{ab}[0,T]}\left(u,\frac{a}{\sqrt{2}}\right)\exp\{i\langle u,y\rangle\}df(u)$$
$$+\frac{i}{2q}\int_{L^2_{ab}[0,T]}[(u,a)(v,a)+(uv,b)]\exp\{i\langle u+v,y\rangle\}df(u)dg(v),$$

$$\int_{C_{ab}[0,T]}^{anf_q}(\delta F(y+x\mid\cdot)*\delta G(y+x\mid\cdot))_q(w)d\mu(x)$$
$$= T_q^{(p)}\left(\delta F\left(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\right)\delta G\left(\cdot \,\Big|\, \frac{w}{\sqrt{2}}\right)\right)(y)$$
$$+\left(\frac{i}{q}\right)^{1/2}\int_{L^2_{ab}[0,T]}\Big[\left\langle u,\frac{w}{\sqrt{2}}\right\rangle\left(v,\frac{a}{\sqrt{2}}\right)-\left(u,\frac{a}{\sqrt{2}}\right)\left\langle v,\frac{w}{\sqrt{2}}\right\rangle$$
$$+\left(\frac{i}{q}\right)^{1/2}\left(u,\frac{a}{\sqrt{2}}\right)\left(v,\frac{a}{\sqrt{2}}\right)+\frac{1}{2}\left(\frac{i}{q}\right)^{1/2}(uv,b)\Big]$$
$$\cdot\exp\left\{i\langle u+v,y\rangle-\frac{i}{2q}((u+v)^2,b)+i\left(\frac{i}{q}\right)^{1/2}(u+v,a)\right\}df(u)dg(v). \tag{4.9}$$

Proof. By using (2.11) and (3.30), we obtain (4.9). □

Next, we obtain the variation of the convolution product of transforms.

Theorem 4.5. *Let F, G, p, q, and w be given as in Theorem 4.1. Then*

$$\delta\left(T_q^{(p)}(F)*T_q^{(p)}(G)\right)_{-q}(y\mid w)$$
$$=\delta T_q^{(p)}\left(F_2\left(\frac{\cdot}{\sqrt{2}}\right)G_2\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y)$$
$$=\int_{L^2_{ab}[0,T]}\frac{i}{\sqrt{2}}\langle u+v,w\rangle \tag{4.10}$$
$$\cdot\exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle-\frac{i}{4q}((u+v)^2,b)\right.$$
$$\left.+i\left(\frac{-i}{2q}\right)^{1/2}(u-v,a)+i\left(\frac{i}{q}\right)^{1/2}(u+v,a)\right\}df(u)dg(v)$$

for s-a.e. $y\in C_{ab}[0,T]$ where F_2 and G_2 are given by (3.19).

Proof. By using (3.17) and the same calculation in the proof of Theorem 3.6, we obtain (4.10). □

Now, we obtain the formulas for convolution product of the variation of the transform. There are two cases; namely, we can take the convolution with respect to the first argument or the second argument of the variation.

Theorem 4.6. *Let F, G, p, q, and w be given as in Theorem 4.1. Then, for s-a.e. $y \in C_{ab}[0,T]$,*

$$\left(\delta T_q^{(p)}(F)(\cdot \mid w) * \delta T_q^{(p)}(G)(\cdot \mid w)\right)_{-q}(y) = T_q^{(p)}\left(\delta F_2\left(\frac{\cdot}{\sqrt{2}} \,\Big|\, w\right)\delta G_2\left(\frac{\cdot}{\sqrt{2}} \,\Big|\, w\right)\right)(y), \tag{4.11}$$

where F_2 and G_2 are given by (3.19); and if $a \in A$, then

$$\begin{aligned}
&\left(\delta T_q^{(p)}(F)(y \mid \cdot) * \delta T_q^{(p)}(G)(y \mid \cdot)\right)_{-q}(w) \\
&\quad = \left(\delta T_q^{(p)}(F)\left(y \,\Big|\, \frac{w}{\sqrt{2}}\right) + \left(-\frac{i}{q}\right)^{1/2}\delta T_q^{(p)}(F)\left(y \,\Big|\, \frac{a}{\sqrt{2}}\right)\right) \\
&\qquad \cdot \left(\delta T_q^{(p)}(G)\left(y \,\Big|\, \frac{w}{\sqrt{2}}\right) - \left(-\frac{i}{q}\right)^{1/2}\delta T_q^{(p)}(G)\left(y \,\Big|\, \frac{a}{\sqrt{2}}\right)\right) \\
&\qquad - \frac{i}{2q}\int_{L_{ab}^2[0,T]}(uv,b)\exp\left\{i\langle u+v,y\rangle - \frac{i}{2q}(u^2+v^2,b)\right. \\
&\qquad\qquad \left. + i\left(\frac{i}{q}\right)^{1/2}(u+v,a)\right\}df(u)dg(v).
\end{aligned} \tag{4.12}$$

Proof. By using (3.22), (3.24), and the same calculation in the proof of Theorem 3.6, we obtain (4.11). Further, proceeding as in the proof of Theorem 3.8 and using (3.33) and (3.22), we have (4.12). □

Theorem 4.7. *Let F, G, p, q, and w be given as in Theorem 4.1 and let $a \in A$. Then, for s-a.e. $y \in C_{ab}[0,T]$,*

$$\begin{aligned}
&\left(T_q^{(p)}(\delta F(y \mid \cdot)) * T_q^{(p)}(\delta G(y \mid \cdot))\right)_q(w) \\
&\quad = (\delta F(y \mid \cdot) * \delta G(y \mid \cdot))_q(w) - \left(\frac{i}{q}\right)^{1/2}\delta F(y \mid a)T_q^{(p)}\left(\delta G\left(y \,\Big|\, \frac{\cdot}{\sqrt{2}}\right)\right)(-w) \\
&\qquad + \left(\frac{i}{q}\right)^{1/2}\delta G(y \mid a)T_q^{(p)}\left(\delta F\left(y \,\Big|\, \frac{\cdot}{\sqrt{2}}\right)\right)(w) + \frac{i}{q}\delta F(y \mid a)\delta G(y \mid a).
\end{aligned} \tag{4.13}$$

Proof. By using (3.24) and a direct calculation, we obtain (4.13). □

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