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STATIONARY POINTS FOR SET-VALUED MAPPINGS ON TWO METRIC SPACES

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Abstract. We give stationary point theorems of set-valued mappings in complete and compact metric spaces. The results in this note generalize a few results due to Fisher.

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1. Introduction and preliminaries. In [2, 4], Fisher and Popa proved fixed point theorems for single-valued mappings on two metric spaces. The purpose of this note is to generalize these results from single-valued mappings into set-valued mappings. In this note, we show stationary point results of set-valued mappings in complete and compact metric spaces.

Let (X,d) and (Y,ρ) be complete metric spaces and $B(X)$ and $B(Y)$ be two families of all nonempty bounded subsets of X and Y, respectively. The function $\delta(A,B)$ with A and B in $B(X)$ is defined as follows:

$$\delta(A,B) = \sup\{d(a,b) : a \in A,\ b \in B\}. \tag{1.1}$$

Define $\delta(A) = \delta(A,A)$. Similarly, the function $\delta'(C,D)$ with C and D in $B(Y)$ is defined as follows:

$$\delta'(C,D) = \sup\{\rho(c,d) : c \in C,\ d \in D\}. \tag{1.2}$$

A sequence of sets in $B(X)$, $\{A_n : n = 1,2,\dots\}$ *converges* to the set A in $B(X)$ if

(i) each point a in A is the limit of some convergent sequence $\{a_n \in A_n : n = 1,2,\dots\}$;

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$, for $n > N$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

Let T be a set-valued mapping of X into $B(X)$. z is a *stationary point* of T if $Tz = \{z\}$. T is *continuous* at x in X if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx in $B(X)$. If T is continuous at each point x in X, then T is a *continuous mapping* of X into $B(X)$.

The following Lemmas 1.1 and 1.2 were proved in [1, 3], respectively.

Lemma 1.1. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X,d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\delta(A,B)$.*

Lemma 1.2. *Let $\{A_n\}$ be a sequence of nonempty subsets of X and let x be a point of X such that $\lim_{n\to\infty}\delta(A_n,x) = 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$.*

2. Stationary point results. Now we prove the following theorem for set-valued mappings.

Theorem 2.1. *Let (X,d) and (Y,ρ) be complete metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the inequalities*

$$\delta(STx,STy) \le c\max\{\delta(x,y),\delta(x,STx),\delta(y,STy),\delta'(Tx,Ty)\}, \tag{2.1}$$

$$\delta'(TSx',TSy') \le c\max\{\delta'(x',y'),\delta'(x',TSx'),\delta'(y',TSy'),\delta(Sx',Sy')\}, \tag{2.2}$$

for all x,y in X and x',y' in Y, where $0 \le c < 1$, then ST has a stationary point z in X and TS has a stationary point w in Y. Further $Tz = \{w\}$ and $Sw = \{z\}$.

Proof. From (2.1) and (2.2), it is easy to see that

$$\begin{gathered}\delta(STA,STB) \le c\max\{\delta(A,B),\delta(A,STA),\delta(B,STB),\delta'(TA,TB)\},\\ \delta'(TSA',TSB') \le c\max\{\delta'(A',B'),\delta'(A',TSA'),\delta'(B',TSB'),\delta(SA',SB')\},\end{gathered} \tag{2.3}$$

for all A,B in $B(X)$ and A',B' in $B(Y)$.

Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$, respectively, by choosing a point x_n in $(ST)^n x = X_n$ and a point y_n in $T(ST)^{n-1}x = Y_n$ for $n = 1,2,\dots$. From (2.3) we have

$$\begin{aligned}\delta(X_n,X_{n+1}) &= \delta(STX_{n-1},STX_n)\\ &\le c\max\{\delta(X_{n-1},X_n),\delta(X_{n-1},X_n),\delta(X_n,X_{n+1}),\delta'(Y_n,Y_{n+1})\}\\ &\le c\max\{\delta(X_{n-1},X_n),\delta'(Y_n,Y_{n+1})\}.\end{aligned} \tag{2.4}$$

Similarly,

$$\delta'(Y_n,Y_{n+1}) \le c\max\{\delta'(Y_{n-1},Y_n),\delta(X_{n-1},X_n)\}. \tag{2.5}$$

Put $M = \max\{\delta(x,X_1),\delta'(Y_1,Y_2)\}$. From the above inequalities, we obtain immediately

$$\delta(X_n,X_{n+1}) \le c^n M, \qquad \delta'(Y_n,Y_{n+1}) \le c^n M, \tag{2.6}$$

for $n \ge 1$. It follows from (2.2) that

$$\begin{aligned}\delta(X_n,X_{n+r}) &\le \delta(X_n,X_{n+1}) + \cdots + \delta(X_{n+r-1},X_{n+r})\\ &\le (c^n + \cdots + c^{n+r-1})M \le \frac{c^n}{1-c}M.\end{aligned} \tag{2.7}$$

Since $c < 1$, then $\delta(X_n,X_{n+r}) \to 0$ as $n \to \infty$. So

$$d(x_n,x_{n+r}) \le \delta(X_n,X_{n+r}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

Thus $\{x_n\}$ is a Cauchy sequence. Completeness of X implies that there exists z in X such that $x_n \to z$ as $n \to \infty$. It follows that

$$\begin{aligned}\delta(z,X_n) &\le \delta(z,x_n) + \delta(x_n,X_n)\\ &\le \delta(z,x_n) + \delta(X_n,X_n)\\ &\le \delta(z,x_n) + 2\delta(X_n,X_{n+1}),\end{aligned} \tag{2.9}$$

which implies that $\delta(z,X_n) \to 0$ as $n \to \infty$. Similarly, there exists w in Y such that $y_n \to w$ and $\delta'(w,Y_n) \to 0$ as $n \to \infty$. Then

$$\delta'(w,Tx_n) \le \delta'(w,TX_n) = \delta'(w,Y_{n+1}). \tag{2.10}$$

By the continuity of T and Lemma 1.1, we have $\delta'(w,Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. Note that

$$\begin{aligned}\delta(STz,x_n) &\le \delta(STz,X_n)\\ &\le c\max\{\delta(z,X_{n-1}),\delta(z,STz),\delta(X_{n-1},X_n),\delta'(Tz,TX_{n-1})\}.\end{aligned} \tag{2.11}$$

Letting n tend to infinity, we have

$$\delta(STz,z) \le c\max\{\delta(STz,z),0\}, \tag{2.12}$$

which implies that $STz = \{z\} = Sw$. Similarly, we can show that w is a stationary point of TS. This completes the proof of the theorem. □

Theorem 2.2. *Let (X,d) be a complete metric space, and let S and T be continuous mappings of X into $B(X)$ and map bounded set into bounded set. If S and T satisfy the inequalities*

$$\begin{aligned}\delta(STx,STy) \le c\max\{&\delta(x,y),\delta(x,STx),\delta(y,STy),\\ &\delta(x,STy),\delta(y,STx),\delta(Tx,Ty)\},\end{aligned} \tag{2.13}$$

$$\begin{aligned}\delta(TSx,TSy) \le c\max\{&\delta(x,y),\delta(x,TSx),\delta(y,TSy),\\ &\delta(x,TSy),\delta(y,TSx),\delta(Sx,Sy)\},\end{aligned} \tag{2.14}$$

for all x,y in X, where $0 \le c < 1$, then ST has a stationary point z and TS has a stationary point w. Further $Tz = \{w\}$ and $Sw = \{z\}$. If $z = w$, then z is the unique common stationary point of S and T.

Proof. Let x be an arbitrary point in X. Define a sequence of sets $\{X_n\}$ by $T(ST)^{n-1}x = X_{2n-1}$, $(ST)^n x = X_{2n}$ for $n \ge 1$ and $X_0 = \{x\}$.

Now suppose that $\{\delta(X_n)\}$ is unbounded. Then the real-valued sequence $\{a_n\}$ is unbounded, where $a_{2n-1} = \delta(X_{2n-1},X_3)$, $a_{2n} = \delta(X_{2n},X_2)$ for $n \ge 1$ and so there exists an integer k such that

$$a_k > \frac{c}{1-c}\max\{\delta(x,X_2),\delta(X_1,X_3)\}, \tag{2.15}$$

$$a_k > \max\{a_1,\dots,a_{k-1}\}. \tag{2.16}$$

Suppose that k is even. Put $k = 2n$. From (2.15) and (2.16) we have

$$\begin{aligned}c\delta(X_{2r},x) &\le c[\delta(X_{2r},X_2)+\delta(X_2,x)] < \delta(X_{2n},X_2),\\ c\delta(X_{2r-1},X_1) &\le c[\delta(X_{2r-1},X_3)+\delta(X_3,X_1) < \delta(X_{2n},X_2)].\end{aligned} \tag{2.17}$$

That is,

$$\delta(X_{2n},X_2) > c\max\{\delta(X_{2r},x),\delta(X_{2r-1},X_1) : 1 \le r \le n\}. \tag{2.18}$$

We now prove that the following inequality is true for $m \geq 1$:

$$\delta(X_{2n}, X_2) \leq c^m \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r,\ s \leq n,\ 2 \leq r',\ s' \leq n\}. \tag{2.19}$$

From (2.13) we have

$$\begin{aligned}\delta(X_{2n}, X_2) &= \delta(STX_{2n-2}, STx) \\ &\leq c \max\{\delta(X_{2n-2}, x), \delta(X_{2n-2}, X_{2n}), \delta(x, X_2), \\ &\qquad \delta(x, X_{2n}), \delta(X_{2n-2}, X_2), \delta(X_{2n-1}, X_1)\}.\end{aligned} \tag{2.20}$$

It follows from (2.16) and (2.18) that

$$\delta(X_{2n}, X_2) \leq c\delta(X_{2n-2}, X_{2n}). \tag{2.21}$$

Now suppose that (2.19) is true for some m. From (2.13), (2.14), (2.16), and (2.18) we have

$$\begin{aligned}\delta(X_{2n}, X_2) &\leq c^m \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r,\ s \leq n,\ 2 \leq r',\ s' \leq n\} \\ &\leq c^{m+1} \max\{\delta(X_{2r-2}, X_{2s-2}), \delta(X_{2r-2}, X_{2r}), \delta(X_{2s-2}, X_{2s}), \\ &\qquad \delta(X_{2r-2}, X_{2s}), \delta(X_{2s-2}, X_{2r}), \delta(X_{2r-1}, X_{2s-1}), \\ &\qquad \delta(X_{2r'-3}, X_{2s'-3}), \delta(X_{2r'-3}, X_{2r'-1}), \\ &\qquad \delta(X_{2s'-3}, X_{2s'-1}) : 1 \leq r,\ s \leq n,\ 2 \leq r',\ s' \leq n\} \\ &\leq c^{m+1} \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \leq r,\ s \leq n,\ 2 \leq r',\ s' \leq n\}.\end{aligned} \tag{2.22}$$

So (2.19) is true for all $m \geq 1$. Letting m tend to infinity, from (2.16) and (2.18) we have $0 < \delta(X_{2n}, X_2) \leq 0$, which is impossible. Similarly, when k is odd, $2n-1$, say, we also have $0 < \delta(X_{2n-1}, X_3) \leq 0$, which is also impossible. Hence $\{\delta(X_n)\}$ is bounded.

Let $M = \sup\{\delta(X_r, X_s) : r,\ s = 0, 1, 2, \ldots\} < \infty$. For arbitrary $\epsilon > 0$, choose a positive integer N such that $c^N M < \epsilon$. Thus for m, n greater than $2N$ with m and n both even or both odd, from (2.13) and (2.14) we have

$$\begin{aligned}\delta(X_m, X_n) &\leq c \max\{\delta(X_{m-2}, X_{n-2}), \delta(X_{m-2}, X_m), \delta(X_{n-2}, X_n), \\ &\qquad \delta(X_{m-2}, X_n), \delta(X_{n-2}, X_m), \delta(X_{m-1}, X_{n-1})\} \\ &\leq c \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : \\ &\qquad m-2 \leq r,\ r' \leq m,\ n-2 \leq s,\ s' \leq n\} \\ &\leq c^N \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : \\ &\qquad m-2N \leq r,\ r' \leq m,\ n-2N \leq s,\ s' \leq n\} \\ &\leq c^N M < \epsilon.\end{aligned} \tag{2.23}$$

So $\delta(X_{2n})$ and $\delta(X_{2n+1}) \to 0$ as $n \to \infty$. Take a point x_n in X_n for $n \geq 1$. Since $d(x_{2n}, x_{2n+2p}) \leq \delta(X_{2n}, X_{2n+2p}) \to 0$ as $n \to \infty$, hence $\{x_{2n}\}$ is a Cauchy sequence. Completeness of X implies that $\{x_{2n}\}$ has a limit z in X. Note that

$$\delta(z, X_{2n}) \leq \delta(z, x_{2n}) + \delta(x_{2n}, X_{2n}) \leq \delta(z, x_{2n}) + \delta(X_{2n}). \tag{2.24}$$

That is, $\delta(z, X_{2n}) \to 0$ as $n \to \infty$. Similarly $\{x_{2n+1}\}$ converges to some point w in X and $\delta(w, X_{2n+1}) \to 0$ as $n \to \infty$. Since $\delta(w, TX_{2n}) = \delta(w, X_{2n+1})$, by the continuity of T and Lemma 1.1, we have $\delta(w, Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. In view of (2.13), we obtain that

$$\begin{aligned}\delta(STz, x_{2n}) &\le \delta(STz, X_{2n}) \\ &\le c\max\{\delta(z, X_{2n-2}), \delta(z, STz), \delta(X_{2n-2}, X_{2n}), \\ &\qquad \delta(z, X_{2n}), \delta(X_{2n-2}, STz), \delta(Tz, X_{2n-1})\},\end{aligned} \tag{2.25}$$

which implies that

$$\delta(STz, z) \le c\max\{\delta(z, STz), 0\} \tag{2.26}$$

as $n \to \infty$. Since $c < 1$, $\delta(STz, z) = 0$. Therefore $STz = \{z\} = Sw$ and $TSw = Tz = \{w\}$.

Now suppose that $z = w$ and that z' is the second common stationary point of S and T. Using (2.1)

$$\begin{aligned}\delta(z, z') &= \delta(STz, STz') \\ &\le c\max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \\ &\qquad \delta(z', STz), \delta(z, STz'), \delta(Tz, Tz')\} \\ &\le c\delta(z, z').\end{aligned} \tag{2.27}$$

So $z = z'$ and this completes the proof of the theorem. □

Remark 2.3. If we use single-valued mappings in place of set-valued mappings in Theorems 2.1 and 2.2, Theorems 2 and 3 of Fisher [2] can be attained.

Remark 2.4. The following example demonstrates that the continuity of S and T in Theorems 2.1 and 2.2 is necessary.

Example 2.5. Let $X = \{0\} \cup \{1/n : n \ge 1\} = Y$ with the usual metric. Define mappings S, T by $T0 = \{1\}$, $T(1/n) = \{1/2n\}$ for $n \ge 1$ and $S = T$. It is easy to prove that all the conditions of Theorems 2.1 and 2.2 are satisfied except that the mappings S and T are continuous. But ST and TS have no stationary points.

Now we give the following theorem for the compact metric spaces.

Theorem 2.6. *Let (X, d) and (Y, ρ) be compact metric spaces. If T is a continuous mapping of X into $B(Y)$ and S is a continuous mapping of Y into $B(X)$ satisfying the following inequalities:*

$$\delta(STx, STy) < \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}, \tag{2.28}$$

$$\delta'(TSx', TSy') < \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\}, \tag{2.29}$$

for all distinct x, y in X and distinct x', y' in Y, then ST has a stationary point z and TS has a stationary point w. Further $Tz = \{w\}$ and $Sw = \{z\}$.

Proof. Suppose that the right-hand sides of inequalities (2.28) and (2.29) are positive for all distinct x, y in X and distinct x', y' in Y. Define the real-valued function

$f(x,y)$ in $X \times X$ as follows:

$$f(x,y) = \frac{\delta(STx,STy)}{\max\{\delta(x,y),\delta(x,STx),\delta(y,STy),\delta'(Tx,Ty)\}}. \tag{2.30}$$

Since S and T are continuous, f is continuous and achieves the maximum value s on the compact metric space $X \times X$. Inequality (2.28) implies $s < 1$. That is,

$$\delta(STx,STy) \le s\max\{\delta(x,y),\delta(x,STx),\delta(y,STy),\delta'(Tx,Ty)\} \tag{2.31}$$

for all distinct x,y in X. It is obvious that (2.31) is also true for $x = y$. Similarly, there exists $t < 1$ such that

$$\delta'(TSx',TSy') \le t\max\{\delta'(x',y'),\delta'(x',TSx'),\delta'(y',TSy'),\delta(Sx',Sy')\} \tag{2.32}$$

for all x',y' in Y. So Theorem 2.6 follows immediately from Theorem 2.1.

Now suppose that there exist z,z' in X such that

$$\max\{\delta(z,z'),\delta(z,STz),\delta(z',STz'),\delta'(Tz,Tz')\} = 0, \tag{2.33}$$

which implies $\{z\} = \{z'\} = STz$ and $Tz = Tz'$, a singleton, $\{w\}$, say. Therefore we have $STz = sw = \{z\}, TSw = Tz = \{w\}$. If there exist w,w' in Y such that

$$\max\{\delta'(w,w'),\delta'(w,TSw),\delta'(w',TSw'),\delta(Sw,Sw')\} = 0. \tag{2.34}$$

Similarly, we also have $STz = Sw = \{z\}$ and $TSw = Tz = \{w\}$. This completes the proof of the theorem. □

Remark 2.7. Theorem 4 of Fisher [2] is a particular case of our Theorem 2.6 if the set-valued mappings in Theorem 2.6 are replaced by single-valued mappings.

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