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LOOPS EMBEDDED IN GENERALIZED CAYLEY ALGEBRAS OF DIMENSION 2^r, $r \geq 2$

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Abstract. Every Cayley algebra of dimension 2^r, $r \geq 2$, contains an embedded invertible loop of order 2^{r+1} generated by its basis. Such a loop belongs to a class of non-abelian invertible loops that are flexible and power-associative.

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1. Introduction. Every finite-dimensional algebra $\mathbb{A}$ over a field $\mathbb{F}$ can be defined by a *multiplication table* of its basis $E_n = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Such a table can be expressed by a matrix $M_r(E_n) = (m_{ij})$, $i,j = 1,\ldots,n$, called a *multiplication matrix* or ⊗-*matrix*, where $m_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j = \sum_{k=1}^{n} \gamma_{ij}^k \mathbf{e}_k$, $\gamma_{ij}^k \in \mathbb{F}$ are its *structure constants*, and $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \in E_n$. By a suitable choice of structure constants, it is possible to construct algebras with desired properties.

There is a class of real algebras called *Cayley algebras of dimension* $n = 2^r$, where $r \geq 2$ [2]. This class includes the classical *Cayley-Dickson algebras* $\mathbb{H}$ (*quaternions*) and $\mathbb{O}$ (*octonions*) [3] as well as the *sedenions* $\mathbb{S}$. In this note, we show that the basis of such an algebra $\mathbb{A}$ forms a non-abelian invertible loop of order 2^{r+1}, called a *Cayley loop*, that is flexible and power-associative. Moreover, we also indicate how the idea of the ⊗-*matrix* can be used in the construction of special algebraic structures (like the group of Dirac operators in quantum electrodynamics).

2. The ⊗-matrix of a Cayley algebra. Consider the ⊗-matrix $M_3(E_8) = (m_{ij})$ shown in Figure 2.1 which defines the algebra of *Cayley numbers* (or *octonions*) $\mathbb{O}$ [2]. If we separate the sign coefficients (or structure constants) z_{ij} of the entries of M_3 into another matrix $Z_3(E_8)$ (Figure 2.2(a)), then the resulting matrix $S_3(E_8)$ can be seen to be the Cayley table of the Klein group $(E_8, \circ)$ of order $n = 8$ shown in Figure 2.2(b). This group is isomorphic to the group $C_2^3 \equiv C_2 \times C_2 \times C_2$, where C_2 is the cyclic group of order 2.

As shown in Figure 2.1, the ⊗-matrix $M_3(E_8)$ that defines $\mathbb{O}$ has two submatrices $M_2(E_4)$ and $M_1(E_2)$ that define the algebras $\mathbb{H}$ and $\mathbb{C}$, respectively. Similarly (Figure 2.2(b)), the matrix $S_3(E_8)$ that defines the Klein group $(E_8, \circ)$ contains two submatrices $S_2(E_4)$ and $S_1(E_2)$ that define its subgroups $(E_4, \circ)$ and $(E_2, \circ)$.

The decomposition of the ⊗-matrix $M_3(E_8)$ into two other matrices $Z_3(E_8)$ and $S_3(E_8)$, therefore, shows that the algebras $\mathbb{O}$, $\mathbb{H}$, and $\mathbb{C}$ are defined by ⊗-matrices of the form $\mathcal{M}_r(E_s) = \mathcal{Z}_r(E_s) \star \mathcal{S}_r(E_s)$, where $\star$ is *Hadamard multiplication*, that is, if $\mathcal{Z}_r = (z_{ij})$ and $\mathcal{S}_r = (e_{ij})$ are matrices of the same dimension, then their Hadamard product is $\mathcal{M}_r = (m_{ij})$, where $m_{ij} = z_{ij} \cdot e_{ij}$, and $\cdot$ is some binary operation.

$$
\begin{array}{c}
\\ M_{1\_} \\ \\ M_{2\_} \\ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & 5 \\
+5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1
\end{bmatrix}
$$

FIGURE 2.1. The $\otimes$-matrix $M_3(E_8) = (m_{ij})$, where $m_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j = z_{ij}\mathbf{e}_k$ which defines the real division algebra A_3 of dimension $n = 8$ isomorphic to the *Cayley numbers* (or *octonions*) $\mathbb{O}$. To simplify the notation for the entries m_{ij}, we have set $z_{ij}k \equiv z_{ij}\mathbf{e}_k$, where $z_{ij} = \pm 1$ and $k = 1,\dots,8$.

$$
\begin{array}{c}
\\ Z_1- \\ \\ Z_2- \\ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
+ & + & + & + & + & + & + & + \\
+ & - & + & - & + & - & - & + \\
+ & - & - & + & + & + & - & - \\
+ & + & - & - & + & - & + & - \\
+ & - & - & - & - & + & + & + \\
+ & + & - & + & - & - & - & + \\
+ & + & + & - & - & + & - & - \\
+ & - & + & + & - & - & + & -
\end{bmatrix}
$$

(a) $Z_3(E_8)$.

$$
\begin{array}{c}
\\ S_1- \\ \\ S_2- \\ \\ \\ \\ \\
\end{array}
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\
4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\
7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
$$

(b) $S_3(E_8)$.

FIGURE 2.2. Decomposition of the $\otimes$-matrix $M_3(E_8)$ into two special matrices. (a) $Z_3(E_8) = (z_{ij})$, $i,j = 1,\dots,8$, where $(z_{ij}) = \pm 1 \in F$ is a *sign matrix*. (b) $S_3(E_8) = (e_{ij})$, where $e_{ij} = \mathbf{e}_i \circ \mathbf{e}_j = \mathbf{e}_v$, is the *structure matrix* of the Klein group $(E_8;\circ) \approxeq C_2^3$ of order 8. For notational simplicity, we have used $\pm$ to represent ± 1 and the subscript v to represent the element $\mathbf{e}_v \in E_8$.

We now formally define the matrices $\mathcal{Z}$, $\mathcal{S}$, and $\mathcal{M}$ as follows.

DEFINITION 2.1. A *sign matrix* is an $m \times n$ matrix $\mathcal{Z} = (z_{ij})$, where $z_{ij} = +1$ or -1 (or simply $+$ or $-$), for every $i = 1,\dots,m$ and for every $j = 1,\dots,n$.

Thus, an $m \times n$ sign matrix [1] is one whose entries are elements of the number set $F = \{+1,-1\} \in \mathbb{R}$. Therefore, they satisfy the following composition rule:

$$(+1)\cdot(+1) = (-1)\cdot(-1) = +1, \qquad (+1)\cdot(-1) = (-1)\cdot(+1) = -1. \tag{2.1}$$

This rule shows that the number set $F = \{+1,-1\}$ is closed under the operation $\cdot$ of multiplication; they form a group $(F,\cdot)$ isomorphic to the cyclic group C_2 of order 2.

Definition 2.2. Let $(E_s, \circ)$ be a finite binary system (like a quasigroup, group or a loop) of order s, where $E_s = \{e_1, \dots, e_s\}$. The matrix $\mathcal{S}(E_s) = (e_{ij})$, where $e_{ij} = e_i \circ e_j$, is called the *structure matrix* (or *Cayley table*) of $(E_s, \circ)$.

Every finite binary system of order s is completely defined by its structure matrix (or Cayley table) which is a listing of all the s^2 possible binary products of its s elements. In the case of finite quasigroups, loops, and groups, their Cayley tables form Latin squares.

Definition 2.3. Let $(E_s, \circ)$ be a binary system of order s, where $E_s = \{e_i \mid i \in I\}$ and $I = \{1, \dots, s\}$, and let $\mathcal{S}_r(E_s) = (e_{ij})$ be its structure matrix, where $e_{ij} = e_i \circ e_j$ for all $i, j \in I$. Let $\mathcal{Z}_r(E_s) = (z_{ij})$ be a given $s \times s$ $\mathcal{Z}$-matrix. The $s \times s$ matrix

$$\mathcal{M}_r(E_s) = \mathcal{Z}_r(E_s) \star \mathcal{S}_r(E_s) = ([m_r]_{ij}) \tag{2.2}$$

is called the *multiplication matrix* or $\otimes$*-matrix* of E_s, where $\star$ is Hadamard multiplication such that

$$[m_r]_{ij} = z_{ij} \cdot e_{ij} = z_{ij} \cdot (e_i \circ e_j) \equiv e_i \otimes e_j \tag{2.3}$$

for all $i, j \in I$, and the operation $\cdot$ is called *sign multiplication*.

In Definition 2.3 of the $\otimes$-*matrix*, we introduced the operation $\otimes$ in terms of the operation $\cdot$ of *sign multiplication* in the expression $e_i \otimes e_j = z_{ij} \cdot e_{ij}$. This operation $\cdot$ simply attaches a sign z_{ij} ($+$ or $-$) to the left of the symbol e_{ij}. If we take $\mathcal{Z}_r$ to be a sign matrix whose entries z_{ij} are the numbers $+1$ and -1; and $\mathcal{S}_r$ to be a structure matrix whose entries e_{ij} are elements of a set E_s of vectors, then we can take the operation $\cdot$ to be ordinary *scalar multiplication* so that the product $e_i \otimes e_j = z_{ij} \cdot e_{ij}$ will be a vector. This would be the case if $\mathcal{M}_r(E_s)$ is the $\otimes$-matrix of a finite-dimensional real algebra whose basis is E_s. This is exemplified by the octonions which we discussed above.

3. The Cayley loops. It follows from Definition 2.3 that if $\mathcal{M}_r(E_s)$ is the $\otimes$-matrix of a real algebra $\mathbb{A}$, then the operation $\otimes$ is closed over $\mathbb{A}$ but not over E_s because of the sign coefficient z_{ij} in its defining equation $e_i \otimes e_j = z_{ij} \cdot (e_i \circ e_j)$. Thus, if $z_{ij} = -1$, then $-(e_i \circ e_j) \notin E_s$. However, if we take the larger set $\mathcal{E} = \{\pm e_i \mid i \in I\}$ of order $\sigma = 2s$, where $+e_i \equiv (+1)e_i = e_i$ and $-e_i \equiv (-1)e_i$, then the operation $\otimes$ will be closed over $\mathcal{E}$. This means that the system $(\mathcal{E}, \otimes)$ is a groupoid embedded in the algebra $\mathbb{A}$. Such a groupoid will be called a $\otimes$-*system*.

Consider once more the octonion algebra $\mathbb{O}$. This is defined by the $\otimes$-matrix $M_3(E_8) = Z_3(E_s) \star S_3(E_8)$ shown in Figure 2.1. For this case, $(E_8, \circ)$ is the Klein group of order $s = 8$, and the operations $\otimes$, $\cdot$, and the matrix Z_3 satisfy the following basic relations:

$$\begin{gathered} e_i \otimes e_j = z_{ij} \cdot (e_i \circ e_j), \\ -e_i = (-1) \cdot e_i, \quad -1 \in F, \\ +e_i = (+1) \cdot e_i = e_i, \quad +1 \in F, \\ (-e_i) \otimes (+e_j) = (+e_i) \otimes (-e_j) = -(e_i \otimes e_j), \\ (-e_i) \otimes (-e_j) = (+e_i) \otimes (+e_j) = e_i \otimes e_j, \end{gathered} \tag{3.1}$$

for all $i, j \in I$, where $F = \{+1, -1\}$ satisfies (2.1).

Equations (3.2) define the basic properties of the entries of the sign matrix Z_3 while (3.3), on the other hand, define the basic properties of the products elements of E_8 under the operation $\otimes$

$$\begin{aligned} z_{ii} &= -1, \quad \text{whenever } i \geq 2, \\ z_{i1} &= z_{1i} = +1, \quad \forall i, \\ z_{ij} &= -z_{ji}, \quad \text{whenever } i \neq j,\ i,j \geq 2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} e_i \otimes e_i &= e_i^2 = -e_1, \quad \text{if } i \geq 2, \\ e_i \otimes e_1 &= e_1 \otimes e_i = e_i, \quad \forall i, \\ e_i \otimes e_j &= -e_j \otimes e_i, \quad \text{if } i \neq j,\ i,j \geq 2. \end{aligned} \tag{3.3}$$

Any real algebra (like the octonions $\mathbb{O}$ and sedenions $\mathbb{S}$) defined by a $\otimes$-matrix of the form $\mathcal{M}_r(E_s) = \mathfrak{X}_r(E_s) \star \mathcal{S}_r(E_s)$, satisfying (3.1), (3.2), and (3.3), where $(E_s, \circ) \approx C_2^r$ will be called a *Cayley algebra of dimension* $s = 2^r$, $r \geq 2$. In such an algebra, the set $\mathscr{E} = \{\pm e_i \mid i = 1, \dots, s\}$ and the operation $\otimes$ form an embedded non-abelian $\otimes$-system $(\mathscr{E}, \otimes)$ that is an invertible loop (*a loop in which every element has a unique two-sided inverse*), where $\delta_i e_i \otimes \delta_j e_j = (\delta_i \delta_j)[z_{ij} \cdot (e_i \circ e_j)]$ and $\delta_i, \delta_j \in F$. This form of the composition rule is implied by (3.1). In the case of the octonions $\mathbb{O}$, the $\otimes$-system $(\mathscr{E}, \otimes)$, where $\mathscr{E} = \{\pm e_i \mid i = 1, \dots, 8\}$, forms a non-abelian invertible loop of order 16 called the *octonion loop*. In general, we have the following theorem.

Theorem 3.1. *Let $(\mathscr{E}, \otimes)$ be a $\otimes$-system embedded in a Cayley algebra $\mathbb{A}$ of dimension 2^r, $r \geq 2$, where $\mathscr{E} = \{\pm e_i \mid i = 1, \dots, s = 2^r\}$ and $\otimes$ is a binary operation over $\mathscr{E}$ satisfying (3.1), (3.2), and (3.3). Then $(\mathscr{E}, \otimes)$ is a non-abelian invertible loop of order 2^{r+1}.*

Proof. By Definition 2.3 and (3.1), (3.2), and (3.3), it follows that $(\mathscr{E}, \otimes)$ is a non-abelian groupoid of order 2^{r+1} with an identity e_1. Moreover, $(\mathscr{E}, \otimes)$ is invertible, that is, every element $e_x \in \mathscr{E}$ has a unique inverse $e_x^{-1} \in \mathscr{E}$. Thus $e_1^{-1} = e_1$ and $e_x^{-1} = -e_x$ since $e_x \otimes (-e_x) = -e_x \otimes e_x = e_1$ for all $x \geq 2$. Similarly, every element $-e_x \in \mathscr{E}$ has a unique inverse $(-e_x)^{-1} = e_x \in \mathscr{E}$. To prove that $(\mathscr{E}, \otimes)$ is an invertible loop, it is therefore sufficient to show that every linear equation has a unique solution. By (3.1), the product of any two elements in $(\mathscr{E}, \otimes)$ is determined primarily by the product $(e_i \circ e_j)$ in $(E_s, \circ)$. Since $(E_s, \circ)$ is a group, then every linear equation has a unique solution. This, together with (3.1) and (3.3), imply that this is also true for $(\mathscr{E}, \otimes)$. Therefore, $(\mathscr{E}, \otimes)$ is an invertible loop. □

Definition 3.2. A $\otimes$-system $(\mathscr{E}, \otimes)$ satisfying (3.1), (3.2), and (3.3), where $(E_s, \circ) \approx C_2^r$ is the generalized Klein group of order $s = 2^r$, $r \geq 2$, is called a *Cayley loop*.

By definition, every Cayley algebra $\mathbb{A}$ of dimension 2^r is defined by a $\otimes$-matrix satisfying (3.1), (3.2), and (3.3). Therefore, it follows from Theorem 3.1 that its embedded $\otimes$-system $(\mathscr{E}, \otimes)$ is a *Cayley loop*. Thus, the octonion loop generated by the basis of the *octonion algebra* is a Cayley loop. Similarly, the loop generated by the basis of the *sedenion algebra* is also a Cayley loop.

The Cayley loop $(\mathscr{E}, \otimes)$ can be explicitly expressed in terms of the matrix $\mathcal{M}_r(E_s) = ([m_r]_{ij})$ as follows. Let $\mathcal{S}(\mathscr{E})$ be the structure matrix of $(\mathscr{E}, \otimes)$. Partition $\mathcal{S}(\mathscr{E})$ into four blocks $\mathscr{E}_{pq}$, $p,q = 1,2$, and let $\mathscr{E}_{11} = \mathscr{E}_{22} = \mathcal{M}_r(E_s)$ and $\mathscr{E}_{12} = \mathscr{E}_{21} = -\mathcal{M}_r(E_s)$, where $-\mathcal{M}_r(E_s) = (-[m_r]_{ij})$. Then we can simply write $\mathcal{S}(\mathscr{E}) = (\mathscr{E}_{pq})$. The structure matrix $\mathcal{S}(\mathscr{E})$ of $(\mathscr{E},\otimes)$ is shown below in block form in terms of the matrix $\mathcal{M}_r(E_s)$

$$\mathcal{S}(\mathscr{E}) = \begin{bmatrix} \mathscr{E}_{11} = \mathcal{M}_r(E_s) & \mathscr{E}_{12} = -\mathcal{M}_r(E_s) \\ \mathscr{E}_{21} = -\mathcal{M}_r(E_s) & \mathscr{E}_{22} = \mathcal{M}_r(E_s) \end{bmatrix}. \tag{3.4}$$

Every Cayley loop or $\otimes$-system $(\mathscr{E},\otimes)$ can be expressed in this matrix form $\mathcal{S}(\mathscr{E})$. This matrix clearly shows that $(\mathscr{E},\otimes)$ is an invertible loop and it can be used as an alternative proof of Theorem 3.1. Many important invertible loops and groups have this structure.

3.1. Construction of Cayley loops. The foregoing considerations show that we can construct special loops by means of $\otimes$-matrices. As an illustration, consider the case of the 4×4 matrix $M_{2,v}(E_4) = Z_{2,v}(E_4) \star S_2(E_4)$ when $n = 4$ so that $r = 2$.

It can be shown [2] that if $Z_{r,v}$ is any $n \times n$ sign matrix satisfying (3.2), then there are exactly $|Z_{r,v}| = 2^\mu$ matrices of this form, where $\mu = \sum_{i=2}^{n-1}(n-i)$. Since $n = 4$, then we find that $\mu = 3$. Hence there are $|Z_{2,v}| = 2^3 = 8$ possible 4×4 $Z_{2,v}$ matrices so that $v = 1,\dots,8$. These eight sign matrices are shown in Figure 3.1.

$$\underset{Z_{2,1}}{\begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & + \\ + & - & - & - \end{bmatrix}} \underset{Z_{2,2}}{\begin{bmatrix} + & + & + & + \\ + & - & + & + \\ + & - & - & - \\ + & - & + & - \end{bmatrix}} \underset{Z_{2,3}}{\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{bmatrix}} \underset{Z_{2,4}}{\begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & - \\ + & + & + & - \end{bmatrix}}$$

$$\underset{Z_{2,5}}{\begin{bmatrix} + & + & + & + \\ + & - & - & - \\ + & + & - & - \\ + & + & + & - \end{bmatrix}} \underset{Z_{2,6}}{\begin{bmatrix} + & + & + & + \\ + & - & - & - \\ + & + & - & + \\ + & + & - & - \end{bmatrix}} \underset{Z_{2,7}}{\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \end{bmatrix}} \underset{Z_{2,8}}{\begin{bmatrix} + & + & + & + \\ + & - & - & + \\ + & + & - & + \\ + & - & - & - \end{bmatrix}}$$

FIGURE 3.1. Eight possible Z-matrices $Z_{2,v}$ that can be used to form eight matrices $M_{2,v}$ [as shown in Figure 3.2] satisfying (3.2). Note that the matrices in the top row are the transposes of those in the bottom row. Thus, $Z_{2,3}$ and $Z_{2,7}$ are transposes, etc.

Figure 3.1 shows the eight matrices $Z_{2,v}$ which, together with the submatrix S_2 shown in Figure 2.2(b), are used to form the eight $\otimes$-matrices $M_{2,v}$ shown in Figure 3.2. These matrices, in turn, can be used to construct eight Cayley loops of order $\sigma = 8$ whose structure matrices have the form given by (3.4).

It can be shown that the $\otimes$-matrices $M_{2,3}$ and $M_{2,7}$ generate loops both of which are isomorphic to the quaternion group. The other six $\otimes$-matrices, on the other hand, generate *non-associative finite invertible loops* (NAFILs) that are isomorphic to each other.

Although the idea of the $\otimes$-matrix $\mathcal{M}_r(E_s) = \mathscr{Z}_r(E_s) \star \mathcal{S}_r(E_s)$ is based on the multiplication matrix of the Cayley algebras, Definition 2.3 is not restricted to these algebraic systems. Such a matrix can therefore be used to construct not only Cayley loops but also other structures (like the group of Dirac operators in quantum electrodynamics [1]) which we call *ZSM loops*. Starting with a given group $(E_s,\circ)$, new systems can

$$\underset{M_{2,1}}{\begin{bmatrix}1&2&3&4\\2&-1&4&3\\3&-4&-1&2\\4&-3&-2&-1\end{bmatrix}}\underset{M_{2,2}}{\begin{bmatrix}1&2&3&4\\2&-1&4&3\\3&-4&-1&-2\\4&-3&2&-1\end{bmatrix}}\underset{M_{2,3}}{\begin{bmatrix}1&2&3&4\\2&-1&4&-3\\3&-4&-1&2\\4&3&-2&-1\end{bmatrix}}\underset{M_{2,4}}{\begin{bmatrix}1&2&3&4\\2&-1&4&-3\\3&-4&-1&-2\\4&3&2&-1\end{bmatrix}}$$

$$\underset{M_{2,5}}{\begin{bmatrix}1&2&3&4\\2&-1&-4&-3\\3&4&-1&-2\\4&3&2&-1\end{bmatrix}}\underset{M_{2,6}}{\begin{bmatrix}1&2&3&4\\2&-1&-4&-3\\3&4&-1&2\\4&3&-2&-1\end{bmatrix}}\underset{M_{2,7}}{\begin{bmatrix}1&2&3&4\\2&-1&-4&3\\3&4&-1&-2\\4&-3&2&-1\end{bmatrix}}\underset{M_{2,8}}{\begin{bmatrix}1&2&3&4\\2&-1&-4&3\\3&4&-1&2\\4&-3&-2&-1\end{bmatrix}}$$

FIGURE 3.2. Eight $\otimes$-matrices $M_{2,v}$ satisfying (3.3). Note that $M_{2,3}$ and $M_{2,7}$ are transposes and that both generate Cayley loops isomorphic to the quaternion group

be formed by means of $\mathcal{X}$-matrices. The given group is thus the substratum of such a system, while the $\mathcal{X}$-matrix determines its special properties.

3.2. Properties of Cayley loops. Finally, we now prove the important theorem that any Cayley loop $(\mathcal{E},\otimes)$ is flexible and power-associative.

THEOREM 3.3. *Let* $(\mathcal{E},\otimes)$ *be a Cayley loop. Then* $(\mathcal{E},\otimes)$ *is flexible and power-associative.*

PROOF. By Theorem 3.1, $(\mathcal{E},\otimes)$ is a non-abelian invertible loop. To prove that it is flexible, let $e_i,e_j \in \mathcal{E}$. Then the following identity (called the *flexible law*) must be satisfied:

$$e_i \otimes (e_j \otimes e_i) = (e_i \otimes e_j) \otimes e_i \tag{3.5}$$

for all $e_i,e_j \in \mathcal{E}$. Clearly, this is trivially satisfied if $i,j = 1$ and also if e_i and e_j are inverses. By (3.3), if $i \neq j$, $i,j \geq 2$, the left side of this identity can be written as $e_i \otimes (e_j \otimes e_i) = -(e_j \otimes e_i) \otimes e_i$. But $(e_j \otimes e_i) = -(e_i \otimes e_j)$ so that we have $-(e_j \otimes e_i) = (e_i \otimes e_j)$. Therefore, it follows that $e_i \otimes (e_j \otimes e_i) = (e_i \otimes e_j) \otimes e_i$; and hence $(\mathcal{E},\otimes)$ is flexible. To prove that $(\mathcal{E},\otimes)$ is power-associative, we must show that it satisfies the following two equations: $e_i \otimes e_i^2 = e_i^2 \otimes e_i = e_i^3$ and $e_i^3 \otimes e_i = e_i^2 \otimes e_i^2$. Since $(\mathcal{E},\otimes)$ is flexible, the first equation is satisfied. Again, by (3.3), if $i \geq 2$ we have $e_i^2 = -e_1$ so that $e_i^3 = e_i^2 \otimes e_i = -e_1 \otimes e_i = -e_i$. Thus $e_i^3 \otimes e_i = -e_i \otimes e_i = -e_i^2 = e_1$ and $e_i^2 \otimes e_i^2 = (-e_1) \otimes (-e_1) = e_1$. Therefore it follows that $e_i^3 \otimes e_i = e_i^2 \otimes e_i^2$. This proves the theorem. □

If $r = 2$, then there exist two Cayley loops of order $n = 8$, one of which is an NAFIL while the other is a group (the quaternion group). All Cayley loops, whether associative or nonassociative, are non-abelian, flexible, and power-associative.

Some Cayley loops (like the octonion loop) are Moufang, and hence also alternative [2] and IP. Others (like the sedenion loop) are alternative and IP but not Moufang.

Although all basic properties of a generalized Cayley algebra are determined by the embedded Cayley loop generated by its basis, not all properties of the loop are satisfied by the algebra. For instance, the sedenion loop that defines the sedenion algebra is alternative but the sedenion algebra is not.

It is easy to show that the elements $e_1, -e_1$ commute and associate with the elements $e_i \in \mathscr{E}$. This implies that the set $\{e_1, -e_1\}$ is the center of the Cayley loop $(\mathscr{E}, \otimes)$.

It would be interesting to find out if the inner mappings of Cayley loops are automorphisms. This, and other interesting questions, are the subject of our present studies.

4. Summary.

The class of *Cayley algebras of dimension* 2^r, where $r \geq 2$, is a generalization of the classical Cayley-Dickson algebras. Such an algebra is defined in terms of its basis $E_s = \{e_1, \dots, e_s\}$ by a $\otimes$-matrix of the form $\mathcal{M}_r(E_s) = \mathcal{X}_r(E_s) \star \mathcal{S}_r(E_s) = ([m_r]_{ij})$, where $[m_r]_{ij} = e_i \otimes e_j = z_{ij} \cdot e_{ij} = z_{ij} \cdot (e_i \circ e_j)$, in which $\otimes$ satisfies (3.1), (3.2), and (3.3). By forming the set $\mathscr{E} = \{\pm e_i \mid i \in I\}$ of order $\sigma = 2^{r+1}$, we showed that the system $(\mathscr{E}, \otimes)$ is a non-abelian invertible loop, called a *Cayley loop*, that is flexible and power-associative.

Although all properties of a generalized Cayley algebra are determined by its Cayley loop, not all properties of the loop are satisfied by the algebra.

References

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