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# THE ABEL-TYPE TRANSFORMATIONS INTO $G_w$

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ABSTRACT. The Abel-type matrix $A_{\alpha,t}$ was introduced and studied as a mapping into $\ell$ by Lemma (1999). The purpose of this paper is to study these transformations as mappings into $G_w$. The necessary and sufficient conditions for $A_{\alpha,t}$ to be $G_w$-$G_w$ are established. The strength of $A_{\alpha,t}$ in the $G_w$-$G_w$ setting is investigated. Also, it is shown that $A_{\alpha,t}$ is translative in the $G_w$-$G_w$ senses for certain sequences.

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**1. Introduction.** The Abel-type power series method [1], denoted by $A_\alpha$, $\alpha > -1$, is the following sequence-to-function transformation: if

$$\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k x^k \text{ is convergent,} \quad \text{for } 0<x<1,$$
$$\lim_{x\to 1}(1-x)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k x^k = L, \tag{1.1}$$

then we say $u$ is $A_\alpha$-summable to $L$. The matrix analogue of $A_\alpha$ is the $A_{\alpha,t}$ matrix [2] whose $nk$th entry is given by

$$a_{nk} = \binom{k+\alpha}{k}t_n^k(1-t_n)^{\alpha+1}, \tag{1.2}$$

where $0 < t_n < 1$ for all $n$ and $\lim t_n = 1$. Thus, the sequence $u$ is transformed into the sequence $A_{\alpha,t}u$ whose $n$th term is given by

$$(A_{\alpha,t}u)_n = (1-t_n)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k t_n^k. \tag{1.3}$$

The matrix $A_{\alpha,t}$ is called the Abel-type matrix [2]. Throughout, $\alpha > -1$ and $t$ will denote such a sequence: $0 < t_n < 1$ for all $n$, and $\lim t_n = 1$.

**2. Basic notations and definitions.** Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. The sequence $Ax$ is called the $A$-transform of the sequence $x$. If $X$ and $Z$ are sets of complex number sequences, then the matrix $A$ is called an $X$-$Z$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Z$ whenever $u$ is in $X$.

Suppose that $y$ is a complex sequence; then throughout we use the following basic notations and definitions:

$$\begin{gathered}\ell = \left\{ y : \sum_{k=0}^{\infty} |y_k| \text{ is convergent} \right\}, \\ d(A) = \left\{ y : \sum_{k=0}^{\infty} a_{nk} y_k \text{ is convergent for each } n \geq 0 \right\}, \\ \ell(A) = \{ y : Ay \in \ell \}, \\ G_w = \{ y : y_k = O(r^k) \text{ for some } r \in (0,w),\ 0 < w < 1 \}, \\ c(A) = \{ y : y \text{ is summable by } A \}, \\ G_w(A) = \{ y : Ay \in G_w \}, \\ \Delta x_k = x_k - x_{k+1}. \end{gathered} \tag{2.2}$$

**Definition 2.1.** The summability matrix $A$ is said to be $G_w$-translative for a sequence $u$ in $G_w(A)$ provided that each of the sequences $T_u$ and $S_u$ is in $G_w(A)$, where $T_u = \{u_1, u_2, u_3, \ldots\}$ and $S_u = \{0, u_0, u_1, \ldots\}$.

**Definition 2.2.** The matrix $A$ is said to be $G_w$-stronger than the matrix $B$ provided $G_w(B) \subseteq G_w(A)$.

## 3. The main results

**Theorem 3.1.** *The matrix $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix if and only if $(1-t)^{\alpha+1} \in G_w$.*

**Proof.** Suppose that $x \in G_w$, then we show that $Y \in G_w$, where $Y$ is the $A_{\alpha,t}$-transform of the sequence $x$. Since $x \in G$, it follows that $|x_k| \leq M_1 r^k$ for some $r \in (0,w)$ and $M_1 > 0$. Now we have

$$\begin{aligned} |Y_n| &= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|, \\ |Y_n| &\leq (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} |x_k| t_n^k \\ &\leq M_1 (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} r^k t_n^k \\ &\leq M_1 (1-t_n)^{\alpha+1} (1-rt_n)^{-(\alpha+1)} \\ &\leq M_2 (1-t_n)^{\alpha+1}, \quad \text{for some } M_2 > 0. \end{aligned} \tag{3.1}$$

Hence if $(1-t)^{\alpha+1} \in G_w$, then it follows that $Y \in G_w$. Conversely, if $(1-t)^{\alpha+1}$ is not in $G_w$, then the first column of $A_{\alpha,t}$ is not in $G_w$ because $a_{n,0} = t_n(1-t_n)^{\alpha+1}$. Thus, $A_{\alpha,t}$ is not a $G_w$-$G_w$ matrix. □

**Remark 3.2.** In the $G_w$-$G_w$ setting, $A_{\alpha,t}$ being a $G_w$-$G_w$ matrix does not imply that $(1-t) \in G_w$. Also, $(1-t) \in G_w$ does not imply that $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix.

This can be demonstrated as follows.

(1) Let $t_n = 1-(1/3)^n$, $\alpha = 1$, and $w = 1/4$. So, we have $(1-t_n)^{\alpha+1} = (1/9)^n$ and hence $(1-t)^{\alpha+1} \in G_w$. This implies that $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix by Theorem 3.1. But observe that $(1-t)$ is not $G_w$. Hence, $A_{\alpha,t}$ being a $G_w$-$G_w$ matrix does not imply that $(1-t) \in G_w$.

(2) Let $t_n = 1-(1/4)^n$, $\alpha = -1/2$, and $w = 1/3$. Then we have $(1-t) \in G_w$. But note that $(1-t_n)^{\alpha+1} = (1/2)^n$ and hence $(1-t)^{\alpha+1}$ is not in $G_w$. This implies that $A_{\alpha,t}$ is not a $G_w$-$G_w$ matrix by Theorem 3.1. Hence, $(1-t) \in G_w$ does not imply that $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix.

**Corollary 3.3.** (1) *If* $-1 < \alpha \leq 0$, *then* $A_{\alpha,t}$ *is a* $G_w$-$G_w$ *matrix implies that* $(1-t) \in G_w$.

(2) *If* $\alpha > 0$, *then* $(1-t) \in G_w$ *implies that* $A_{\alpha,t}$ *is a* $G_w$-$G_w$ *matrix.*

**Proof.** (1) Since $-1 < \alpha \leq 0$ implies that $(1-t_n) \leq (1-t_n)^{\alpha+1}$, it follows that $(1-t) \in G_w$ by Theorem 3.1.

(2) If $\alpha > 0$, then we have $(1-t_n)^{\alpha+1} < (1-t_n)$ and hence by Theorem 3.1, $A_{\alpha,t}$ a $G_w$-$G_w$ matrix whenever $(1-t) \in G_w$. □

**Corollary 3.4.** *The matrix* $A_{\alpha,t}$ *is a* $G$-$G_w$ *matrix if and only if* $A_{\alpha,t}$ *is a* $G_w$-$G_w$ *matrix.*

**Proof.** Since $G_w$ is a subset of $G, A_{\alpha,t}$ being a $G$-$G_w$ matrix yields $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix. Conversely, if $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix, then by Theorem 3.1, we have $(1-t)^{\alpha+1} \in G_w$. Now using the same technique used in the proof of Theorem 3.1, we can easily show that $A_{\alpha,t}$ is a $G$-$G_w$ matrix. Thus, the corollary follows. □

The next results indicate that the $A_{\alpha,t}$ matrix is a strong method in the $G_w$-$G_w$ setting. The $A_{\alpha,t}$ matrix is $G_w$-stronger than the identity matrix.

**Theorem 3.5.** *Suppose that* $-1 < \alpha \leq 0$ *and* $A_{\alpha,t}$ *is a* $G_w$-$G_w$ *matrix; then* $G_w(A_{\alpha,t})$ *contains the class of all sequences* $x$ *whose partial sums are bounded.*

**Proof.** The theorem follows using a similar argument as in the proof of [2, Theorem 8]. □

**Remark 3.6.** Although Theorem 3.5 is stated for $-1 < \alpha \leq 0$, it is also true for all $\alpha > -1$ for some sequences, which we will demonstrate as follows. Let $x$ be the unbounded sequence defined by

$$x_k = (-1)^k \frac{k+\alpha+1}{\alpha+1}. \tag{3.2}$$

Let $Y$ be the $A_{\alpha,t}$-transform of $x$. Then we have

$$Y_n = \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+2}} < (1-t_n)^{\alpha+1}. \tag{3.3}$$

Thus, if $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix, then by Theorem 3.1, $(1-t)^{\alpha+1} \in G_w$, so $x \in G_w(A_{\alpha,t})$.

**Corollary 3.7.** *Suppose that $-1 < \alpha \le 0$ and $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix; then $G_w(A_{\alpha,t})$ contains the class of all sequences $x$ such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent.*

Our next results deal with the $G_w$-translativity of the $A_{\alpha,t}$ matrix. We will show that the $A_{\alpha,t}$ matrix is $G_w$-translative for some sequences in $G_w(A_{\alpha,t})$.

**Theorem 3.8.** *Every $G_w$-$G_w A_{\alpha,t}$ matrix is $G_w$-translative for each sequence $x \in G_w(A_{\alpha,t})$ for which $\{x_k/k\} \in G_w$, $k = 1,2,3,\dots$.*

**Proof.** Let $x \in G_w(A_{\alpha,t})$. Then we will show that

(1) $T_x \in G_w(A_{\alpha,t})$ and

(2) $S_x \in G_w(A_{\alpha,t})$.

We first show that (1) holds. Note that

$$\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k \frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k \left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\le A_n + B_n,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
A_n &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|, \\
B_n &= \frac{|\alpha|(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|.
\end{aligned} \tag{3.5}$$

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show both $A$ and $B$ are in $G_w$, then (1) holds. But the conditions that $A \in G_w$ and $B \in G_w$ follow easily from the given hypothesis that $x \in G_w(A_{\alpha,t})$ and $\{x_k/k\} \in G_w$, respectively.

Next we will show that (2) holds. Observe that

$$\begin{aligned}\left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right|\\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right|\\ &\le E_n+F_n,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}E_n &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|,\\ F_n &= (1-t_n)^{\alpha+1}|\alpha|\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+}\right|.\end{aligned} \tag{3.7}$$

Now the given hypothesis that $x \in G_w(A_{\alpha,t})$ and $\{x_k/k\} \in G_w$ implies that both $E$ and $F$ are in $G_w$. Consequently, (2) holds and hence the theorem follows. □

**Theorem 3.9.** *Suppose that $-1 < \alpha \le 0$; then every $G_w$-$G_w$ matrix $A_{\alpha,t}$ is $G_w$-translative for each $A_\alpha$-summable sequence $x$ in $G_w(A_{\alpha,t})$.*

**Proof.** Since the case $\alpha = 0$ can be easily proved using the technique used in the proof of [4, Theorem 4.1], here we only consider the case $-1 < \alpha < 0$. Let $x \in c(A_\alpha)\cap G_w(A_{\alpha,t})$. Then we will show that

(1) $T_x \in G_w(A_{\alpha,t})$ and
(2) $S_x \in G_w(A_{\alpha,t})$.

We first show that (1) holds. Note that

$$\begin{aligned}\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right|\\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right|\\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right|\\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{k}{k+\alpha}\right|\\ &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right|\\ &\le A_n+B_n,\end{aligned} \tag{3.8}$$

where

$$
\begin{aligned}
A_n &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|, \\
B_n &= -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|.
\end{aligned}
\tag{3.9}
$$

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show that both $A$ and $B$ are in $G_w$, then (1) holds. The condition $A \in G_w$ follows from the hypothesis that $x \in G_w(A_{\alpha,t})$, and $B \in G_w$ will be shown as follows. Observe that

$$
\begin{aligned}
B_n &= -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|x_1t_n+\sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right| \\
&\le -\alpha|x_1|(1-t_n)^{\alpha+1}+\frac{-\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right| \\
&\le C_n+D_n,
\end{aligned}
\tag{3.10}
$$

where

$$
\begin{aligned}
C_n &= -\alpha|x_1|(1-t_n)^{\alpha+1}, \\
D_n &= -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|.
\end{aligned}
\tag{3.11}
$$

By Theorem 3.1, the hypothesis that $A_{\alpha,t}$ is $G_w$-$G_w$ implies that $C \in G_w$, hence there remains only to show $D \in G_w$ to prove that (1) holds. Now using the same techniques used in the proof of [3, Theorem 2], we can show that

$$
D_n \le \frac{M_1M_2}{\alpha}(1-t_n)-\frac{M_1M_2}{\alpha}(1-t_n)^{\alpha+1},
\tag{3.12}
$$

where $M_1$ and $M_2$ are some positive real numbers. Note that $A_{\alpha,t}$ being a $G_w$-$G_w$ matrix implies that $(1-t)^{\alpha+1} \in G_w$ by Theorem 3.1, and $-1 < \alpha < 0$ yields $(1-t) \in G_w$. Consequently, we have $D \in G_w$ and hence (1) holds. Next we show that (2) holds. We have

$$
\begin{aligned}
\left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right| \\
&\le E_n+F_n,
\end{aligned}
\tag{3.13}
$$

where

$$E_n = (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|,$$
$$F_n = -(1-t_n)^{\alpha+1} \alpha \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} \frac{x_k}{k+1} t_n^{k+} \right|. \tag{3.14}$$

The hypothesis that $x \in G_w(A_{\alpha,t})$ implies that $E \in G_w$ and by proceeding as in the proof of (1) above, we can easily show that $F \in G_w$. Thus, (2) holds and hence our assertion follows. □

**Theorem 3.10.** *Suppose that $\alpha > 0$ and $(1-t) \in G_w$; then every $A_{\alpha,t}$ matrix is $G_w$-translative for each $A_\alpha$-summable sequence $x$ in $G_w(A_{\alpha,t})$.*

**Proof.** The theorem follows easily by using similar argument used in the proof of Theorem 3.9. □

Our next result is a Tauberian theorem for $A_{\alpha,t}$ matrix in the $G_w$-$G_w$ setting.

**Theorem 3.11.** *Let $A_{\alpha,t}$ be a $G_w$-$G_w$ matrix. If $x$ is a sequence such that $A_{\alpha,t}x$ and $\Delta x$ are in $G_w$, then $x$ is in $G_w$.*

**Proof.** The theorem easily follows by an argument similar to the proof of [4, Theorem 2.1]. □

## References


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