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SOME APPLICATIONS OF MINIMAL OPEN SETS

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ABSTRACT. We characterize minimal open sets in topological spaces. We show that any nonempty subset of a minimal open set is pre-open. As an application of a theory of minimal open sets, we obtain a sufficient condition for a locally finite space to be a pre-Hausdorff space.

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1. Introduction. Let X be a topological space. We call a nonempty open set U of X a minimal open set when the only open subsets of U are U and $\emptyset$.

In this paper, we study fundamental properties of minimal open sets and apply them to obtain some results on pre-open sets (cf. [2]) and pre-Hausdorff spaces.

In Section 2, we characterize minimal open sets, that is, we show that a nonempty open set U is a minimal open set if and only if $\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U. This result implies that any nonempty subset S of a minimal open set U is a pre-open set.

In Section 3, we study minimal open sets in locally finite spaces. The results of this section are closely related to the work of James [1], and these results will be used in the next scetion.

In Section 4, we apply the theory of minimal open sets to study pre-open sets. Our first main result of this section is a property of the set of all minimal open sets in any nonempty finite open set which is not a minimal open set. This result enables us to prove a generalization of Theorem 2.5, when U is a nonempty finite open set, in Theorem 4.4. Theorem 4.5 shows that our theory of minimal open set is useful to study pre-open sets.

Finally, we show that some conditions on minimal open sets implies pre-Hausdorffness of a space, that is, if any minimal open set of a locally finite space X has two elements at least, then X is a pre-Hausdorff space.

2. Minimal open sets. Let (X,τ) be a topological space.

DEFINITION 2.1. A nonempty open set U of X is said to be a minimal open set if and only if any open set which is contained in U is $\emptyset$ or U.

LEMMA 2.2. (1) *Let U be a minimal open set and W an open set. Then $U \cap W = \emptyset$ or $U \subset W$.*

(2) *Let U and V be minimal open sets. Then $U \cap V = \emptyset$ or $U = V$.*

Proof. (1) Let W be an open set such that $U \cap W \neq \varnothing$. Since U is a minimal open set and $U \cap W \subset U$, we have $U \cap W = U$. Therefore $U \subset W$.

(2) If $U \cap V \neq \varnothing$, then we see that $U \subset V$ and $V \subset U$ by (1). Therefore $U = V$. □

Proposition 2.3. *Let U be a minimal open set. If x is an element of U, then $U \subset W$ for any open neighborhood W of x.*

Proof. Let W be an open neighborhood of x such that $U \not\subset W$. Then $U \cap W$ is an open set such that $U \cap W \subsetneq U$ and $U \cap W \neq \varnothing$. This contradicts our assumption that U is a minimal open set. □

Proposition 2.4. *Let U be a minimal open set. Then*

$$U = \cap\{W \mid W \text{ is an open neighborhood of } x\} \tag{2.1}$$

for any element x of U.

Proof. By Proposition 2.3 and the fact that U is an open neighborhood of x, we have $U \subset \cap\{W \mid W$ is an open neighborhood of $x\} \subset U$. Therefore we have the result. □

Theorem 2.5. *Let U be a nonempty open set. Then the following three conditions are equivalent:*

(1) *U is a minimal open set.*

(2) *$U \subset \mathrm{Cl}(S)$ for any nonempty subset S of U.*

(3) *$\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U.*

Proof. (1)⇒(2). Let S be any nonempty subset of U. By Proposition 2.3, for any element x of U and any open neighborhood W of x, we have

$$S = U \cap S \subset W \cap S. \tag{2.2}$$

Then, we have $W \cap S \neq \varnothing$ and hence x is an element of $\mathrm{Cl}(S)$. It follows that $U \subset \mathrm{Cl}(S)$.

(2)⇒(3). For any nonempty subset S of U, we have $\mathrm{Cl}(S) \subset \mathrm{Cl}(U)$. On the other hand, by (2), we see $\mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Cl}(S)) = \mathrm{Cl}(S)$. Therefore we have $\mathrm{Cl}(U) = \mathrm{Cl}(S)$ for any nonempty subset S of U.

(3)⇒(1). Suppose that U is not a minimal open set. Then there exists a nonempty open set V such that $V \subsetneq U$ and hence there exists an element $a \in U$ such that $a \notin V$. Then we have $\mathrm{Cl}(\{a\}) \subset V^c$, the complement of V. It follows that $\mathrm{Cl}(\{a\}) \neq \mathrm{Cl}(U)$. □

A subset M of a space (X,τ) is called a *pre-open* set if $M \subset \mathrm{Int}\,\mathrm{Cl}(M)$. The family of all pre-open sets in (X,τ) will be denoted by $\mathrm{PO}(X,\tau)$, (cf. [2]).

A space (X,τ) is called *pre-Hausdorff* if for each $x, y \in X$, $x \neq y$ there exist subsets $U, V \in \mathrm{PO}(X,\tau)$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Theorem 2.6. *Let U be a minimal open set. Then any nonempty subset S of U is a pre-open set.*

Proof. By Theorem 2.5(2), we have $\mathrm{Int}\,U \subset \mathrm{Int}\,\mathrm{Cl}(S)$. Since U is an open set, we have $S \subset U = \mathrm{Int}(U) \subset \mathrm{Int}\,\mathrm{Cl}(S)$. □

Theorem 2.7. *Let U be a minimal open set and M a nonempty subset of X. If there exists an open neighborhood W of M such that $W \subset \mathrm{Cl}(M \cup U)$, then $M \cup S$ is a pre-open set for any nonempty subset S of U.*

Proof. By Theorem 2.5(3), we have $\mathrm{Cl}(M \cup S) = \mathrm{Cl}(M) \cup \mathrm{Cl}(S) = \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. Since $W \subset \mathrm{Cl}(M \cup U) = \mathrm{Cl}(M \cup S)$ by assumption, we have $\mathrm{Int}(W) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Since W is an open neighborhood of M, namely W is an open set such that $M \subset W$, we have $M \subset W = \mathrm{Int}(W) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Moreover we have $\mathrm{Int}(U) \subset \mathrm{Int}\,\mathrm{Cl}(M \cup U)$, for $\mathrm{Int}(U) = U \subset \mathrm{Cl}(U) \subset \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. Since U is an open set, we have $S \subset U = \mathrm{Int}\,U \subset \mathrm{Int}\,\mathrm{Cl}(M \cup U) = \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. Therefore $M \cup S \subset \mathrm{Int}\,\mathrm{Cl}(M \cup S)$. □

Corollary 2.8. *Let U be a minimal open set and M a nonempty subset of X. If there exists an open neighborhood W of M such that $W \subset \mathrm{Cl}(U)$, then $M \cup S$ is a pre-open set for any nonempty subset S of U.*

Proof. By assumption, we have $W \subset \mathrm{Cl}(M) \cup \mathrm{Cl}(U) = \mathrm{Cl}(M \cup U)$. So by Theorem 2.7, we see that $M \cup S$ is a pre-open set. □

The condition of Theorem 2.7, namely $W \subset \mathrm{Cl}(M \cup S)$, does not necessarily imply the condition of Corollary 2.8, namely $W \subset \mathrm{Cl}(S)$. We have the following example.

Example 2.9. Let $X = \{a,b,c,d\}$ with topology $\theta = \{\emptyset, \{d\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$, $U = \{a,b\}$ and $M = W = \{d\}$. Then $W = \{d\} \subset \mathrm{Cl}(\{a,b\} \cup \{d\}) = \mathrm{Cl}(M \cup U)$ and $W = \{d\} \not\subset \mathrm{Cl}(\{a,b\}) = \mathrm{Cl}(U)$.

Theorem 2.10. *Let U be a minimal open set and x an element of $X - U$. Then $W \cap U = \emptyset$ or $U \subset W$ for any open neighborhood W of x.*

Proof. Since W is an open set, we have the result by Lemma 2.2. □

Corollary 2.11. *Let U be a minimal open set and x an element of $X - U$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then $U_x \cap U = \emptyset$ or $U \subset U_x$.*

Proof. If $U \subset W$ for any open neighborhood W of x, then $U \subset \cap\{W \mid W$ is an open neighborhood of $x\}$. Therefore $U \subset U_x$. Otherwise there exists an open neighborhood W of x such that $W \cap U = \emptyset$. Then we have $U \cap U_x = \emptyset$. □

3. Finite open sets.

In this section, we study some properties of minimal open sets in finite open sets and locally finite spaces.

Theorem 3.1. *Let V be a nonempty finite open set. Then there exists at least one (finite) minimal open set U such that $U \subset V$.*

Proof. If V is a minimal open set, we may set $U = V$. If V is not a minimal open set, then there exists an (finite) open set V_1 such that $\emptyset \neq V_1 \subsetneq V$. If V_1 is a minimal open set, we may set $U = V_1$. If V_1 is not a minimal open set, then there exists an (finite) open set V_2 such that $\emptyset \neq V_2 \subsetneq V_1 \subsetneq V$. Continuing this process, we have a sequence of open sets

$$V \supsetneq V_1 \supsetneq V_2 \cdots \supsetneq V_k \supsetneq \cdots. \tag{3.1}$$

Since V is a finite set, this process repeats only finitely. Then, finally we get a minimal open set $U = V_n$ for some positive integer n. □

A topological space is said to be a *locally finite space* if each of its elements is contained in a finite open set.

Corollary 3.2. *Let X be a locally finite space and V a nonempty open set. Then there exists at least one (finite) minimal open set U such that $U \subset V$.*

Proof. Since V is a nonempty set, there exists an element x of V. Since X is a locally finite space, we have a finite open set V_x such that $x \in V_x$. Since $V \cap V_x$ is a finite open set, we get a minimal open set U such that $U \subset V \cap V_x \subset V$ by Theorem 3.1. □

Theorem 3.3. *Let V_λ be an open set for any $\lambda \in \Lambda$ and W a nonempty finite open set. Then $W \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is a finite open set.*

Proof. We see that there exists an integer n such that $W \cap (\cap_{\lambda\in\Lambda} V_\lambda) = W \cap (\cap_{i=1}^{n} V_{\lambda_i})$ and hence we have the result. □

Theorem 3.4. *Let V_λ be an open set for any $\lambda \in \Lambda$ and W_μ a nonempty finite open set for any $\mu \in \mathcal{M}$. Let $S = \cup_{\mu\in\mathcal{M}} W_\mu$. Then $S \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is an open set.*

Proof. Since W_μ is a finite open set, by Theorem 3.3, we have $W_\mu \cap (\cap_{\lambda\in\Lambda} V_\lambda)$ is a finite open set for any $\mu \in \mathcal{M}$. Since

$$S \cap (\cap_{\lambda\in\Lambda} V_\lambda) = (\cup_{\mu\in\mathcal{M}} W_\mu) \cap (\cap_{\lambda\in\Lambda} V_\lambda) = \cup_{\mu\in\mathcal{M}} (W_\mu \cap (\cap_{\lambda\in\Lambda} V_\lambda)), \tag{3.2}$$

we have the result. □

Corollary 3.5 (see [1]). *Any locally finite space is an Alexandroff space.*

4. Applications. Let U be a nonempty finite open set. We see, by Lemma 2.2 and Corollary 3.2, that there exists a positive integer k such that $\{U_1, U_2, \dots, U_k\}$ is the set of all minimal open sets in U. Then it satisfies the following two conditions:

(a) $U_i \cap U_j = \emptyset$ for any i, j with $1 \le i, j \le k$, and $i \ne j$.

(b) If U' is a minimal open set in U, then there exists i with $1 \le i \le k$ such that $U' = U_i$.

Theorem 4.1. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset U_x$.*

Proof. Assume that $U_i \not\subset U_x$ for any positive integer i of $\{1, \dots, n\}$. Then we have $U_i \cap U_x = \emptyset$ for any minimal open set U_i in U by Corollary 2.11. Since U_x is a nonempty finite open set by Theorem 3.3, there exists a minimal open set U' such that $U' \subset U_x$ by Theorem 3.1. Since $U' \subset U_x \subset U$, we have U' is a minimal open set in U. By assumption, we have $U_i \cap U' \subset U_i \cap U_x = \emptyset$ for any minimal open set U_i. Therefore $U' \ne U_i$ for any positive integer i of $\{1, 2, \dots, n\}$. This contradicts our assumption. □

Proposition 4.2. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset W_x$ for any open neighborhood W_x of x.*

Proof. Since $W_x \supset \cap \{W \mid W$ is an open neighborhood of $x\}$, we have the result by Theorem 4.1. □

Theorem 4.3. *Let U be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \dots, U_n\}$ be the set of all minimal open sets in U and x an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer i of $\{1, \dots, n\}$ such that x is an element of $\mathrm{Cl}(U_i)$.*

Proof. By Proposition 4.2, there exists a positive integer i of $\{1, \dots, n\}$ such that $U_i \subset W$ for any open neighborhood W of x. Therefore $U_i \cap W \supset U_i \cap U_i \neq \emptyset$ for any open neighborhood W of x. Therefore we have the result. □

The following result is a generalization of Theorem 2.5, when U is a nonempty finite open set.

Theorem 4.4. *Let U be a nonempty finite open set and U_i a minimal open set in U for each $i \in \{1, 2, \dots, n\}$. Then the following three conditions are equivalent:*

(1) *$\{U_1, U_2, \dots, U_n\}$ is the set of all minimal open sets in U.*

(2) *$U \subset \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets S_i of U_i for $i \in \{1, 2, \dots, n\}$.*

(3) *$\mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets S_i of U_i for $i \in \{1, 2, \dots, n\}$.*

Proof. (1)⇒(2). If U is a minimal open set, then this is the result of Theorem 2.5(2). Otherwise U is not a minimal open set. If x is any element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$, we have $x \in \mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_2) \cup \cdots \cup \mathrm{Cl}(U_n)$ by Theorem 4.3. Therefore

$$
\begin{aligned}
U \subset \mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_2) \cup \cdots \cup \mathrm{Cl}(U_n) &= \mathrm{Cl}(S_1) \cup \mathrm{Cl}(S_2) \cup \cdots \cup \mathrm{Cl}(S_n) \\
&= \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)
\end{aligned}
\tag{4.1}
$$

by Theorem 2.5(3).

(2)⇒(3). For any nonempty subset S_i of U_i with $i \in \{1, 2, \dots, n\}$, we have $\mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n) \subset \mathrm{Cl}(U)$. On the other hand, by (2), we see

$$
\mathrm{Cl}(U) \subset \mathrm{Cl}(\mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n). \tag{4.2}
$$

Therefore we have $\mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subset S_i of U_i with $i \in \{1, 2, \dots, n\}$.

(3)⇒(1). Suppose that V is a minimal open set in U and $V \neq U_i$ for $i \in \{1, 2, \dots, n\}$. Then we have $V \cap \mathrm{Cl}(U_i) = \emptyset$ for each $i \in \{1, 2, \dots, n\}$. It follows that any element of V is not contained in $\mathrm{Cl}(U_1 \cup U_2 \cup \cdots \cup U_n)$. This contradicts the condition (3) because $V \subset U \subset \mathrm{Cl}(U) = \mathrm{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$. □

Let U be a nonempty finite open set, $\{U_1, U_2, \dots, U_n\}$ the set of all minimal open sets in U and x_i an element of U_i for each $i \in \{1, 2, \dots, n\}$. Then we see that the set $\{x_1, x_2, \dots, x_n\}$ is a pre-open set by Theorem 4.4. Moreover, we have the following result.

Theorem 4.5. *Let U be a nonempty finite open set and $\{U_1, U_2, \ldots, U_n\}$ the set of all minimal open sets in U. Let S be any subset of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$ and S_i be any nonempty subset of U_i for each $i \in \{1, 2, \ldots, n\}$. Then $S \cup S_1 \cup S_2 \cdots \cup S_n$ is a pre-open set.*

Proof. By Theorem 4.4(2), we have

$$U \subset \mathrm{Cl}\,(S_1 \cup S_2 \cdots \cup S_n) \subset \mathrm{Cl}\,(S \cup S_1 \cup S_2 \cdots \cup S_n). \tag{4.3}$$

Since U is an open set, then we have

$$S \cup S_1 \cup S_2 \cdots \cup S_n \subset U = \mathrm{Int}(U) \subset \mathrm{Int}\,\mathrm{Cl}\,(S \cup S_1 \cup S_2 \cdots \cup S_n). \tag{4.4}$$

Then we have the result. □

Theorem 4.6. *Let X be a locally finite space. If any minimal open set of X has two elements at least, then X is a pre-Hausdorff space.*

Proof. Let x, y be elements of X such that $x \neq y$. Since X is a locally finite space, there exists finite open sets U and V such that $x \in U$ and $y \in V$. By Theorem 3.1, there exists the set $\{U_1, U_2, \ldots, U_n\}$ of all minimal open sets in U and the set $\{V_1, V_2, \ldots, V_m\}$ of all minimal open sets in V.

Case 1. If there exists i of $\{1, 2, \ldots, n\}$ and j of $\{1, 2, \ldots, m\}$ such that $x \in U_i$ and $y \in V_j$, then, by Theorem 2.6, $\{x\}$ and $\{y\}$ are disjoint pre-open sets which contains x and y, respectively.

Case 2. If there exists i of $\{1, 2, \ldots, n\}$ such that $x \in U_i$ and $y \notin V_j$ for any j of $\{1, 2, \ldots, m\}$, then we find an element y_j of V_j for each j such that $\{x\}$ and $\{y, y_1, y_2, \ldots, y_m\}$ are pre-open sets and $\{x\} \cap \{y, y_1, y_2, \ldots, y_n\} = \varnothing$ by Theorems 2.6, 4.5 and the assumption.

Case 3. If $x \notin U_i$ for any i of $\{1, 2, \ldots, n\}$ and $y \notin V_j$ for any j of $\{1, 2, \ldots, m\}$, then we find elements x_i of U_i and y_j of V_j for each i, j such that $\{x, x_1, x_2, \ldots, x_n\}$ and $\{y, y_1, y_2, \ldots, y_m\}$ are pre-open sets and $\{x, x_1, x_2, \ldots, x_n\} \cap \{y, y_1, y_2, \ldots, y_m\} = \varnothing$ by Theorem 4.5 and the assumption. We remark that we use the assumption that any minimal open set of X has at least two elements for the case $U_i = V_j$ for some i and j in the argument of cases (2) and (3).

Therefore X is a pre-Hausdorff space. □

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