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ROUGH MARCINKIEWICZ INTEGRAL OPERATORS

HUSSAIN AL-QASSEM and AHMAD AL-SALMAN

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ABSTRACT. We study the Marcinkiewicz integral operator $M_{\mathcal{P}}f(x) = (\int_{-\infty}^{\infty} |\int_{|y|\le 2^t} f(x - \mathcal{P}(y))(\Omega(y)/|y|^{n-1})\,dy|^2\,dt/2^{2t})^{1/2}$, where $\mathcal{P}$ is a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$ and Ω is a homogeneous function of degree zero on $\mathbb{R}^n$ with mean value zero over the unit sphere $\mathbf{S}^{n-1}$. We prove an L^p boundedness result of $M_{\mathcal{P}}$ for rough Ω.

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1. Introduction. Let $\mathbb{R}^n$, $n \ge 2$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure. Consider the Marcinkiewics integral operator

$$\mu f(x) = \left(\int_{-\infty}^{\infty} |\mathbf{F}_t(x)|^2 \frac{dt}{2^{2t}}\right)^{1/2}, \tag{1.1}$$

where

$$\mathbf{F}_t(x) = \int_{|x-y|\le 2^t} f(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}\,dy, \tag{1.2}$$

and Ω is a homogeneous function of degree zero which has the following properties:

$$\Omega \in L^1(\mathbf{S}^{n-1}), \qquad \int_{\mathbf{S}^{n-1}} \Omega(y')\,d\sigma(y') = 0. \tag{1.3}$$

When $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \le 1)$, Stein proved the L^p boundedness of $\mu(f)$ for all $1 < p \le 2$. Subsequently, Benedek, Calderón, and Panzone proved the L^p boundedness of $\mu(f)$ for all $1 < p < \infty$ under the condition $\Omega \in C^1(\mathbf{S}^{n-1})$ (see [2]).

The authors of [3] were able to prove the following result for the more general class of operators

$$\mu_P f(x) = \left(\int_{-\infty}^{\infty} |\mathbf{F}_{P,t}(x)|^2 \frac{dt}{2^{2t}}\right)^{1/2}, \tag{1.4}$$

where

$$\mathbf{F}_{P,t}(x) = \int_{|y|\le 2^t} f(x - P(|y|)y')\frac{\Omega(y)}{|y|^{n-1}}\,dy \tag{1.5}$$

and P is a real-valued polynomial on $\mathbb{R}$ and satisfies $P(0) = 0$.

Theorem 1.1 (see [3]). *Let $\alpha > 0$, and $\Omega \in V_\alpha(n)$. Then the operator μ_P is bounded in $L^p(\mathbb{R}^n)$ for $(2\alpha+2)/(2\alpha+1) < p < 2+2\alpha$.*

In [1], Al-Salman and Pan studied the singular integral operator

$$\mathbf{T}_{\Omega,\mathcal{P}} f(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x - \mathcal{P}(y)) \frac{\Omega(y')}{|y|^n} \, dy, \tag{1.6}$$

where $\mathcal{P} = (P_1, \dots, P_d)$: $\mathbb{R}^n \to \mathbb{R}^d$ is a polynomial mapping, $d \geq 1$, $n \geq 2$. The authors of [1] proved that $\mathbf{T}_{\Omega,\mathcal{P}}$ is bounded in $L^p(\mathbb{R}^d)$ whenever $(2+2\alpha)/(1+2\alpha) < p < 2+2\alpha$ and $\Omega \in W_\alpha(n)$. Here $W_\alpha(n)$ is a subspace of $L^1(\mathbf{S}^{n-1})$ and its definition as well as the definition of $V_\alpha(n)$ will be reviewed in Section 2. It was shown in [1] that $W_\alpha(n) = V_\alpha(n)$, *if* $n = 2$ and it is a proper subspace of $V_\alpha(n)$ if $n \geq 3$.

Our purpose in this paper is to study the L^p boundedness of the class of operators

$$M_{\mathcal{P}} f(x) = \left(\int_{-\infty}^{\infty} |\mathbf{F}_{\mathcal{P},t}(x)|^2 \frac{dt}{2^{2t}} \right)^{1/2}, \tag{1.7}$$

where

$$\mathbf{F}_{\mathcal{P},t}(x) = \int_{|y| \leq 2^t} f(x - \mathcal{P}(y)) \frac{\Omega(y)}{|y|^{n-1}} \, dy. \tag{1.8}$$

Our main result in this paper is the following theorem.

Theorem 1.2. *Let $\alpha > 0$, and $\Omega \in W_\alpha(n)$. Then the operator $M_{\mathcal{P}}$ is bounded in $L^p(\mathbb{R}^d)$ for $(2\alpha+2)/(2\alpha+1) < p < 2+2\alpha$. The bound of $M_{\mathcal{P}} f$ is independent of the coefficients of $\{P_j\}$.*

By [1, Theorem 3.1] and Theorem 1.2 we have the following corollary.

Corollary 1.3. *Let $\alpha > 0$, $\Omega \in V_\alpha(2)$ and $\mathcal{P} : R^2 \to R^d$. Then $M_{\mathcal{P}}$ is bounded in $L^p(R^d)$ for $(2\alpha+2)/(2\alpha+1) < p < 2+2\alpha$. The bound of $M_{\mathcal{P}}$ is independent of the coefficients of $\{P_j\}$.*

2. Preparation.

We start this section by recalling the following definition from [1].

Definition 2.1. For $\alpha > 0$, $N \geq 1$, let $\widetilde{\mathcal{V}}(n,N) = \bigcup_{m=1}^{N} \mathcal{V}(n,m)$ and let $W_\alpha(N,n)$ be the subspace of $L^1(\mathbf{S}^{n-1})$ defined by

$$W_\alpha(N,n) = \left\{ \Omega \in L^1(\mathbf{S}^{n-1}) : \int_{\mathbf{S}^{n-1}} \Omega(y') \, d\sigma(y') = 0, \; M_\alpha(N,n) < \infty \right\}, \tag{2.1}$$

where

$$M_\alpha(N,n) = \max\left\{ \int_{\mathbf{S}^{n-1}} |\Omega(y')| \left(\log \frac{1}{|P(y')|} \right)^{1+\alpha} d\sigma(y') : P \in \widetilde{\mathcal{V}}(n,N) \text{ with } \|P\| = 1 \right\}. \tag{2.2}$$

For $\alpha > 0$, we define $W_\alpha(n)$ to be

$$W_\alpha(n) = \bigcap_{N=1}^{\infty} W_\alpha(N,n). \tag{2.3}$$

Also, for $\alpha > 0$, we define $V_\alpha(n)$ by $V_\alpha(n) = W_\alpha(1,n)$ (see [6]).

Here $\mathcal{V}(n,m)$ is the space of all real-valued homogeneous polynomials on $\mathbb{R}^n$ with degree equal to m and with norm $\|\cdot\|$ defined by

$$\left\| \sum_{|\alpha|=m} a_\alpha y^\alpha \right\| = \sum_{|\alpha|=m} |a_\alpha|. \tag{2.4}$$

Now we need to recall the following results.

Lemma 2.2 (see van der Corput [7]). *Suppose ϕ and ψ are real-valued and smooth in (a,b), and that $|\phi^{(k)}(t)| \ge 1$ for all $t \in (a,b)$. Then the inequality*

$$\left| \int_a^b e^{-i\lambda\phi(t)}\psi(t)\,dt \right| \le C_k|\lambda|^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(t)|\,dt\right], \tag{2.5}$$

holds when
(i) $k \ge 2$, *or*
(ii) $k = 1$ *and ϕ' is monotonic.*
The bound C_k is independent of a, b, ϕ, and λ.

Lemma 2.3 (see [7]). *Let $\mathcal{P} = (P_1,\dots,P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$. Let $\deg(\mathcal{P}) = \max_{1\le j\le d}\deg(P_j)$. Suppose $\Omega \in L^1(\mathbf{S}^{n-1})$ and*

$$\mu_{\Omega,\mathcal{P}}f(x) = \sup_{h>0}\left| \frac{1}{h^n}\int_{|y|<h} f(x-\mathcal{P}(y))\Omega(y')\,dy \right|. \tag{2.6}$$

Then for every $1 < p \le \infty$, there exists a constant $C_p > 0$ which is independent of Ω and the coefficients of $\{P_j\}$ such that

$$\|\mu_{\Omega,\mathcal{P}}f\|_p \le C_p\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\|f\|_p \tag{2.7}$$

for every $f \in L^p(\mathbb{R}^d)$.

To each polynomial mapping $\mathcal{P} = (P_1,\dots,P_d) : \mathbb{R}^n \to \mathbb{R}^d$ with

$$\deg\mathcal{P} = \max_{1\le j\le d}\deg P_j = N, \quad d \ge 1,\ n \ge 2, \tag{2.8}$$

we define a family of measures

$$\left\{\vartheta_t^l, \lambda_t^l : l = 0,1,\dots,N,\ t \in \mathbb{R}\right\} \tag{2.9}$$

as follows.

For $1 \le j \le d$, $0 \le l \le N$ let $P_j = \sum_{|\alpha|\le N} C_{j\alpha}y^\alpha$ and let $Q^l = (Q_1^l,\dots,Q_d^l)$ where $Q_j^l = \sum_{|\alpha|\le l} C_{j\alpha}y^\alpha$.

Now for $0 \le l \le N$ and $t \in \mathbb{R}$, let ϑ_t^l and λ_t^l be the measures defined in the Fourier transform side by

$$\begin{aligned} (\vartheta_t^l)\hat{}(\xi) &= \int_{|y|\le 2^t} e^{-2\pi i\xi\cdot Q^l(y)}\frac{\Omega(y')}{|y|^{n-1}}\frac{dy}{2^t}, \\ (\lambda_t^l)\hat{}(\xi) &= \int_{|y|\le 2^t} e^{-2\pi i\xi\cdot Q^l(y)}\frac{|\Omega(y')|}{|y|^{n-1}}\frac{dy}{2^t}. \end{aligned} \tag{2.10}$$

The maximal functions $(\vartheta^l)^*$ defined by

$$(\vartheta^l)^*(f)(x) = \sup_{t\in\mathbb{R}} |\lambda_t^l * f(x)|, \tag{2.11}$$

for $l = 0,1,\dots,N$.

For later purposes, we need the following definition.

Definition 2.4. For each $1 \le l \le N$, let $N_l = |\{\alpha \in \mathbf{N}^n : |\alpha| = l\}|$ and let $\{\alpha \in \mathbf{N}^n : |\alpha| = l\} = \{\alpha_1,\dots,\alpha_{N_l}\}$. For each $1 \le l \le N$, define the linear transformations $L_l^{\alpha_j} : \mathbb{R}^d \to \mathbb{R}$ and $L_l : \mathbb{R}^d \to \mathbb{R}^{N_l}$ by

$$\begin{aligned} L_l^{\alpha_j}(\xi) &= \sum_{i=1}^{d} \left(C_{i,\alpha_j y^{\alpha_j}}\right)\xi_i, \quad j = 1,\dots,N_l, \\ L_l(\xi) &= \left(L_l^{\alpha_1}(\xi),\dots,L_l^{\alpha_{N_l}}(\xi)\right). \end{aligned} \tag{2.12}$$

To simplify the proof of our result we need the following lemma.

Lemma 2.5. *Let $\{\sigma_t^l : l = 0,1,\dots,N,\ t \in \mathbb{R}\}$ be a family of measures such that $\sigma_t^0 = 0$ for all $t \in \mathbb{R}$. Let $\mathbf{D}_l : \mathbb{R}^n \to \mathbb{R}^d$, $l = 0,1,\dots,N$ be linear transformations. Suppose that for all $t \in \mathbb{R}$ and $l = 0,1,\dots,N$, then*

$$\begin{aligned} &\|\sigma_t^l\| \le C(l), \\ &|(\sigma_t^l)\hat{}(\xi)| \le C\frac{M_\alpha}{(\log[c2^{lt}|D_l(\xi)|])^{1+\alpha}}, \\ &|(\sigma_t^l)\hat{}(\xi) - (\sigma_t^{l-1})\hat{}(\xi)| \le C2^{lt}|D_l(\xi)|. \end{aligned} \tag{2.13}$$

Then there exists a family of measures $\{\nu_t^l : l = 1,\dots,N\}_{t\in\mathbb{R}}$ such that

$$\begin{aligned} &\|\nu_t^l\| \le C(l), \\ &|(\nu_t^l)\hat{}(\xi)| \le C\frac{M_\alpha}{(\log[c2^{lt}|D_l(\xi)|])^{1+\alpha}}, \\ &|(\nu_t^l)\hat{}(\xi)| \le C2^{lt}|D_l(\xi)|, \\ &\sigma_t^N = \sum_{l=1}^{N} \nu_t^l. \end{aligned} \tag{2.14}$$

Proof. By [5, Lemma 6.1], for each $l = 1,\dots,N$ choose two nonsingular linear transformations

$$A_l : \mathbb{R}^{r(l)} \longrightarrow \mathbb{R}^d, \qquad B_l : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \tag{2.15}$$

such that

$$|A_l\pi_{r(l)}^d B_l(\xi)| \le |D_l(\xi)| \le N|A_l\pi_{r(l)}^d B_l(\xi)|, \quad \xi \in \mathbb{R}^d, \tag{2.16}$$

where $r(l) = \operatorname{rank}(D_l)$ and $\pi_{r(l)}^d$ is the projection operator from $\mathbb{R}^d$ into $\mathbb{R}^{r(l)}$.

Now choose $\eta \in C_0^\infty(\mathbb{R})$ such that $\eta(t) = 1$ for $|t| \le 1/2$ and $\eta(t) = 0$ for $|t| \ge 1$. Let $\varphi(t) = \phi(t^2)$ and let

$$(\nu_t^l)\hat{}(\xi) = (\sigma_t^l)\hat{}(\xi) \prod_{l<j\le N} \varphi(|2^{tj} A_j \pi_{r(j)}^d B_j(\xi)|) - (\sigma_t^{l-1})\hat{}(\xi) \prod_{l-1<j\le N} \varphi(|2^{tj} A_j \pi_{r(j)}^d B_j(\xi)|) \tag{2.17}$$

with the convention $\prod_{j\in\emptyset} a_j = 1$, $1 \le l \le N$.

Hence, one can easily see that $\{\sigma_t^l : l = 1,\dots,N,\ t \in \mathbb{R}\}$ is the desired family of measures. □

Now for the boundedness of the maximal functions $(\vartheta^l)^*$, $l = 0,1,\dots,N$, we have the following lemma whose proof is an easy consequence of Lemma 2.3, polar coordinates and Hölder's inequality:

Lemma 2.6. *For $l = 1,\dots,N$ and $p \in (1,\infty)$, there exists a constant $C_{p,l}$ which is independent of the coefficients of the polynomial components of the mapping Q^l such that*

$$||(\vartheta^l)^* f||_p \le C_{p,l} \|f\|_p. \tag{2.18}$$

3. Boundedness of some square functions. For a nonnegative C^∞ radial function Φ on $\mathbb{R}^n$ with

$$\operatorname{supp}(\Phi) \subset \left\{x \in \mathbb{R}^n : \frac{1}{2} \le |x| \le 2\right\}, \qquad \int_0^\infty \frac{\Phi(t)}{t}\, dt = 1, \tag{3.1}$$

and for a linear transformation $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^d$, define the functions ψ_t, $t \in \mathbb{R}$ by $\hat{\psi}_t(y) = \Phi(2^t \mathbf{L}(y))$.

For a family of measures $\{\sigma_t\}_{t\in\mathbb{R}}$, real number u and $l \in \mathbb{N}$, let $\mathbf{J}_u^l(f)$ be the square function defined by

$$\mathbf{J}_u^l(f)(x) = \left(\int_{-\infty}^\infty |\sigma_t * \psi_{l(t+u)} * f(x)|^2\, dt\right)^{1/2}. \tag{3.2}$$

For such a square function we have the following theorem.

Theorem 3.1. *If $\{\sigma_t\}_{t\in\mathbb{R}}$ is a family of measures such that the corresponding maximal function*

$$\sigma^*(f)(x) = \sup_{t\in\mathbb{R}} ||\sigma_t| * f(x)| \tag{3.3}$$

is bounded on $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$, then

$$||\mathbf{J}_u^l(f)||_{L^p(\mathbb{R}^d)} \le C_{p,l} \sqrt{||\sigma^*||_{(p/2)'} \sup_{t\in\mathbb{R}} ||\sigma_t||}\, ||f||_{L^p(\mathbb{R}^d)} \tag{3.4}$$

for every $1 < p < \infty$. Here $C_{p,l}$ is a constant that depends only on p and the dimension of the underlying space.

Proof. If $\sup_{t\in\mathbb{R}}\|\sigma_t\| = \infty$, then the inequality holds trivially. Thus we may assume that $\sup_{t\in\mathbb{R}}\|\sigma_t\| < \infty$. In this case we follow a similar argument as in [4]. Let $p > 2$ and $q = (p/2)'$. Choose a nonnegative function $v \in L^q_+$ with $\|v\|_q = 1$ such that

$$||\mathbf{J}^l_u(f)||_p^2 = \int_{\mathbb{R}^d}\left(\int_{-\infty}^{\infty} |\sigma_t * \psi_{l(t+u)} * f(x)|^2 dt\right) v(x)dx. \tag{3.5}$$

Thus it is easy to see that

$$\begin{aligned} ||\mathbf{J}^l_u(f)||_p^2 &\le \sup_{t\in\mathbb{R}}||\sigma_t|| \int_{-\infty}^{\infty}\int_{\mathbb{R}^d} |\psi_{l(t+u)} * f(z)|^2 \sigma^*(v)(-z)dz\,dt \\ &\le \sup_{t\in\mathbb{R}}||\sigma_t|| \int_{\mathbb{R}^d} [g(f)]^2(z)\sigma^*(v)(-z)\,dz, \end{aligned} \tag{3.6}$$

where

$$g(f)(x) = \left(\int_{-\infty}^{\infty} |\psi_{l(t+u)} * f(x)|^2 dt\right)^{1/2}. \tag{3.7}$$

Now since $\int_{\mathbb{R}^d}\psi_t(x)\,dx = 0$, it is well known that

$$\|g(f)\|_p \le C_p\|f\|_p \quad \forall 1 < p < \infty \tag{3.8}$$

with constant C_p that depends only on p and the dimension of the underlying space.

Thus by (3.6) and Hölder's inequality we have

$$\begin{aligned} ||\mathbf{J}^l_u(f)||_p^2 &\le \sup_{t\in\mathbb{R}}||\sigma_t||\,\|g(f)\|_p^2||\sigma^*(u)||_q \\ &\le C_p^2 \sup_{t\in\mathbb{R}}||\sigma_t||\,||\sigma^*||_{(p/2)'}\|f\|_p^2. \end{aligned} \tag{3.9}$$

Hence our result follows by taking the square root on both sides. The case $p < 2$ follows by duality. □

4. Proof of the main theorem. Let $\alpha > 0$, $\Omega \in W_\alpha(n)$. Let $\mathcal{P} = (P_1,\dots,P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$ with $\deg\mathcal{P} = \max_{1\le j\le d}\deg P_j = N$, where $d \ge 1$ and $n \ge 2$. For $0 \le l \le N$ let N_l, Q^l, v^l_t, λ^l_t, and L_l be as in Section 3.

The first step in our proof is to show that each ϑ^l_t, $l = 1,\dots,N$ satisfies the hypotheses of Lemma 2.5, that is,

$$||\vartheta^l_t|| \le C(l), \tag{4.1}$$

$$|(\vartheta^l_t)\hat{}(\xi)| \le C\frac{M_\alpha}{(\log[c2^{lt}|L_l(\xi)|])^{1+\alpha}}, \tag{4.2}$$

$$|(\vartheta^l_t)\hat{}(\xi) - (\vartheta^{l-1}_t)\hat{}(\xi)| \le C2^{lt}|L_l(\xi)|. \tag{4.3}$$

One can easily see that (4.1) holds trivially. Using the cancellation property of Ω, it is easy to see that (4.3) holds. Thus, we need only to verify (4.2). To see that, we notice that

$$|(\vartheta^l_t)\hat{}(\xi)| \le \int_{S^{n-1}}|\Omega(y')|\left|\int_0^1 e^{-2\pi i\xi\cdot Q^l(2^t r y')}dr\right|d\sigma(y'). \tag{4.4}$$

Now the quantity $\xi \cdot Q^l(2^{tl} r y')$ can be written in the form

$$\xi \cdot Q^l(2^{tl} r y') = 2^{tl} r^l \lambda G^l(y') + \xi \cdot R(2^t r y'), \tag{4.5}$$

where Q^l is a homogeneous polynomial of degree l with $\|G^l\| = 1$, R is a polynomial of degree at most $l-1$ in the variable r,

$$\lambda = \sum_{j=1}^{N_l} |L_l^{\alpha_j}(\xi)| \geq N_l |L_l(\xi)| \tag{4.6}$$

and $\alpha_1, \ldots, \alpha_{N_l}$ are the constants that appeared in Section 2. Thus by van der Corput lemma, we have

$$\left| \int_0^1 e^{-2\pi i \xi \cdot Q^l(2^t r y')} dr \right| \leq C \min \left\{ 1, \left(2^{tl} |L_l(\xi)| \, |G^l(y')|\right)^{-1/l} \right\} \tag{4.7}$$

and hence

$$\left| \int_0^1 e^{-2\pi i \xi \cdot Q^l(2^t r y')} dr \right| \leq C \frac{[\log(c|G^l(y')|^{-1})]^{1+\alpha}}{(\log[c 2^{tl}|L_l(\xi)|])^{1+\alpha}}, \tag{4.8}$$

where C is a constant independent of t and ξ. Since $\Omega \in W_\alpha(n)$, the estimate (4.2) follows.

By Lemma 2.5, there exists a family of measures $\{\nu_t^l : l = 1, \ldots, N, \ t \in \mathbb{R}\}$ such that

$$\|\nu_t^l\| \leq C(l), \tag{4.9}$$

$$|(\hat{\nu_t^l})(\xi)| \leq C \frac{M_\alpha}{(\log[c 2^{lt}|L_l(\xi)|])^{1+\alpha}}, \tag{4.10}$$

$$|(\hat{\nu_t^l})(\xi)| \leq C 2^{lt} |L_l(\xi)|, \tag{4.11}$$

$$\vartheta_t^N = \sum_{l=1}^N \nu_t^l. \tag{4.12}$$

Also by Lemma 2.6 and the definition of ν_t^l (see the proof of Lemma 2.5), we have

$$\|(\nu^l)^* f\|_p \leq C_{p,l} \|f\|_p \quad \forall 1 < p < \infty. \tag{4.13}$$

Now one can easily see that

$$2^{-t} \mathbf{F}_{\mathcal{P},t}(x) = \vartheta_t^N * f(x) = \sum_{l=1}^N \nu_t^l * f(x). \tag{4.14}$$

Therefore,

$$\|M_{\mathcal{P}} f\|_p \leq \sum_{l=1}^N \|M_{\mathcal{P}}^l f\|_p, \tag{4.15}$$

where

$$M_{\mathcal{P}}^l f(x) = \left(\int_{-\infty}^{\infty} |\nu_t^l * f(x)|^2 dt \right)^{1/2}. \tag{4.16}$$

Thus to show the boundedness of $M_{\mathcal{P}}f$, it suffices to show that

$$||M_{\mathcal{P}}^l f||_p \le C_{p,l}\|f\|_p \tag{4.17}$$

for $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$, and for all $l = 1,\dots,N$.

To show (4.17), we proceed as follows: let Φ and ψ_t be as in Section 3. Then

$$\begin{aligned} M_{\mathcal{P}}^l f(x) &= \log 2^l \left(\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \nu_t^l * \psi_{l(t+u)} * f(x)\, du \right|^2 dt \right)^{1/2} \\ &\le \log 2^l \int_{-\infty}^{\infty} S_u^l f(x)\, du, \end{aligned} \tag{4.18}$$

where

$$S_u^l f(x) = \left(\int_{-\infty}^{\infty} |\nu_t^l * \psi_{l(t+u)} * f(x)|^2 dt \right)^{1/2}. \tag{4.19}$$

Now by (4.13) and Theorem 3.1, we have

$$||S_u^l f||_p \le C_p \|f\|_p \tag{4.20}$$

for all $p \in (1,\infty)$ and for $l = 1,\dots,N$ which in turn implies that

$$\int_{-1}^{1} ||S_u^l f||_p\, du \le 2C_p \|f\|_p \quad \forall p \in (1,\infty). \tag{4.21}$$

On the other hand, if $u \ge 1$, by the estimate (4.11) we have

$$\begin{aligned} ||S_u^l f||_2^2 &= \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |\nu_t^l * \psi_{l(t+u)} * f(x)|^2 dt\, dx \\ &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (\Phi(2^{lt+lu}\mathbf{L}_l(\xi)))^2 |(\nu_t^l)\hat{}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi\, dt \\ &\le 2^{2l-2lu} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \left(\int_{\log(1/2^l|\mathbf{L}_l(\xi)|)-u}^{\log(2^l/|\mathbf{L}_l(\xi)|)-u} dt \right) d\xi \\ &= 2\log 2^l 2^{2l-2lu} \|f\|_2^2. \end{aligned} \tag{4.22}$$

Thus

$$||S_u^l f||_2 \le \sqrt{2\log 2^l}\, 2^{l-lu} \|f\|_2. \tag{4.23}$$

By interpolating between (4.20) and (4.23) we get

$$||S_u^l f||_p \le C_{p,l} 2^{\theta l - \theta l u} \|f\|_p \tag{4.24}$$

for all $1 < p < \infty$ and for some $\theta = \theta(p) > 0$. Hence we have

$$\int_{1}^{\infty} ||S_u^l f||_p\, du \le C_p \|f\|_p \quad \text{for } p \in (1,\infty). \tag{4.25}$$

Finally, if $u < -1$, by the estimate (4.10) and similar argument as in the case of $u \le 1$, we get

$$||S_u^l f||_2 \le C_l(|u|)^{-1-\alpha}\|f\|_2. \tag{4.26}$$

By interpolating between (4.26) and any $p \in (1,\infty)$ in (4.20), we get that, if $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$ there exists $\beta > 0$ such that

$$||S_u^l f||_p \le C_p(|u|)^{-\beta}\|f\|_p, \tag{4.27}$$

which implies that

$$\int_{-\infty}^{-1} ||S_u^l f||_p \, du \le C_p\|f\|_p \tag{4.28}$$

for $p \in ((2+2\alpha)/(1+2\alpha), 2+2\alpha)$.

Hence by combining (4.18), (4.21), (4.25), and (4.28) we get (4.17). □

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Hussain Al-Qassem: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: husseink@yu.edu.jo

Ahmad Al-Salman: Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail address: alsalman@yu.edu.jo