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EVALUATION OF EULER-ZAGIER SUMS

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ABSTRACT. We present a simple method for evaluation of multiple Euler sums in terms of single and double zeta values.

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1. Introduction.

We give a short evaluation of the triple sums

$$w(p,q,r) = \sum_{n,m=1}^{\infty} \frac{1}{n^p m^q (n+m)^r} \tag{1.1}$$

in terms of single zeta values $\zeta(p)$

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} \tag{1.2}$$

and double zeta values (Euler sums) $S(p,q)$

$$S(p,q) = \sum_{n=1}^{\infty} H_n^{(p)} n^{-q}, \quad H_n^{(p)} = 1^{-p} + 2^{-p} + \cdots + n^{-p}, \tag{1.3}$$

where $p \geq 1$, $q > 1$.

Multiple Euler sums have been discussed and evaluated in a number of papers of which we want to point out [1, 2, 3, 4, 5, 6, 7, 10]. Also [8, Sections 18 and 19]. We refer to these publications for general comments and details.

2. Euler sums

Lemma 2.1. *For any integer $p > 1$ and any $x > 0$,*

$$\sigma(p;x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p(n+x)} = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}\zeta(p-k+1)}{x^k} + \frac{(-1)^{p-1}}{x^p}(\psi(x+1)+\gamma), \tag{2.1}$$

where $\psi = \Gamma'/\Gamma$ is the psi function and γ is Euler's constant.

Proof. We have

$$\begin{aligned} \sigma(p;x) &= \sum_{n=1}^{\infty} \frac{x+n-n}{n^p x(n+x)} = \sum_{n=1}^{\infty} \frac{1}{n^p x} - \sum_{n=1}^{\infty} \frac{1}{n^{p-1}x(n+x)} \\ &= \frac{1}{x}(\zeta(p) - \sigma(p-1;x)). \end{aligned} \tag{2.2}$$

Equation (2.1) follows by repeating the procedure $p-1$ times in view of the fact that (see [9, page 665])

$$\sigma(1;x) = \sum_{n=1}^{\infty} \frac{1}{n(n+x)} = \frac{1}{x}(\psi(x+1)+\gamma). \tag{2.3}$$

Now we differentiate (2.1) $r-1$ times, where $r > 1$. With $D = d/dx$ we have

$$D^{r-1}\frac{1}{n+x} = \frac{(-1)^{r-1}(r-1)!}{(n+x)^r},$$

$$D^{r-1}\frac{1}{x^k} = (-1)^{r-1}\frac{(r+k-2)!}{(k-1)!}\frac{1}{x^{r+k-1}} = (-1)^{r-1}(r-1)!\binom{r+k-2}{r-1}\frac{1}{x^{r+k-1}}, \tag{2.4}$$

$$D^{r-1}(x^{-p}(\psi(x+1)+\gamma)) = \sum_{k=0}^{r-1}\binom{r-1}{k}(D^{r-1-k}x^{-p})(D^k(\psi(x+1)+\gamma)).$$

Therefore,

$$\begin{aligned} D^{r-1}\sigma(p;x) &= (-1)^{r-1}(r-1)!\sum_{n=1}^{\infty}\frac{1}{n^p(n+x)^r} \\ &= (-1)^{r-1}(r-1)!\sum_{k=1}^{p-1}(-1)^{k-1}\binom{r+k-2}{r-1}\frac{\zeta(p-k+1)}{x^{k+r-1}} \\ &\quad + \frac{(-1)^{p-1}(-1)^{r-1}(r-1)!}{(p-1)!}\sum_{k=0}^{r-1}\frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!}\frac{(\psi(x+1)+\gamma)^{(k)}}{x^{r+p-k-1}}. \end{aligned} \tag{2.5}$$

□

We summarize this result in the following lemma.

Lemma 2.2. *For any integers $p > 1$, $r \ge 1$ and any $x > 0$,*

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^p(n+x)^r} &= \sum_{k=1}^{p-1}(-1)^{k-1}\binom{r+k-2}{r-1}\frac{\zeta(p-k+1)}{x^{k+r-1}} \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-1}\frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!}\frac{(\psi(x+1)+\gamma)^{(k)}}{x^{r+p-k-1}}. \end{aligned} \tag{2.6}$$

Next, we replace here x by mx and multiply both sides by m^{-q}, $q \ge 1$. This gives

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^p m^q(n+mx)^r} &= \sum_{k=0}^{p-2}\frac{(-1)^k}{x^{k+r}}\binom{r+k-1}{r-1}\frac{\zeta(p-k)}{m^{k+q+r}} \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-1}\frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!}\frac{1}{x^{r+p-k-1}}\frac{(\psi(mx+1)+\gamma)^{(k)}}{m^{r+p+q-k-1}}. \end{aligned} \tag{2.7}$$

Summing for $m = 1,2,\ldots$ we obtain our main representation.

Theorem 2.3. *For all integers $p > 1$, $r \geq 1$ and all $q \geq 0$, $q+r > 1$, $x > 0$,*

$$\begin{aligned}\sigma(p,q,r;x) &\equiv \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^p m^q (n+mx)^r}\\ &= \sum_{k=0}^{p-2}\frac{(-1)^k}{x^{k+r}}\binom{r+k-1}{r-1}\zeta(p-k)\zeta(k+q+r)\\ &+\frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-1}\frac{(-1)^k(r+p-k-2)!}{k!(r-k-1)!}\frac{1}{x^{r+p-k-1}}\sum_{m=1}^{\infty}\frac{(\psi(mx+1)+\gamma)^{(k)}}{m^{r+p+q-k-1}}.\end{aligned} \tag{2.8}$$

The case $p = 1$ can be derived directly from (2.3), namely,

$$\begin{aligned}\sigma(1,q,r;x) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{nm^q(n+mx)^r}\\ &= \sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\frac{1}{x^{r-k}}\sum_{m=1}^{\infty}\frac{(\psi(mx+1)+\gamma)^{(k)}}{m^{r+q-k}}.\end{aligned} \tag{2.9}$$

We remind the reader that the expression $(\psi(mx+1)+\gamma)^{(k)}$ stands for the kth derivative of the function $\psi(x+1)+\gamma$ evaluated at mx.

By setting $x = 1$ we get the desired representation of $w(p,q,r)$. Making use of

$$\begin{aligned}\psi(m+1)+\gamma &= H_m^{(1)} = 1+2^{-1}+\cdots+m^{-1},\\ \psi^{(k)}(m+1) &= (-1)^k k![H_m^{(k+1)}-\zeta(k+1)],\end{aligned} \tag{2.10}$$

(see [9, page 775]), and with the agreement to read $\zeta(1) = 0$, one obtains the following corollary.

Corollary 2.4. *For all integers $p > 1$, $r \geq 1$ and all $q \geq 0$ with $q+r > 1$,*

$$\begin{aligned}w(p,q,r) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^p m^q (n+m)^r}\\ &= \sum_{k=0}^{p-2}(-1)^k\binom{r+k-1}{r-1}\zeta(p-k)\zeta(r+q+k)\\ &+\frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-1}\frac{(r+p-k-2)!}{(r-k-1)!}[S(k+1,r+p+q-k-1)-\zeta(k+1)\zeta(r+p+q-k-1)],\end{aligned} \tag{2.11}$$

in particular,

$$\begin{aligned}w(1,q,r) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{nm^q(n+m)^r}\\ &= \sum_{k=0}^{r-1}[S(k+1,r+q-k)-\zeta(k+1)\zeta(r+q-k)].\end{aligned} \tag{2.12}$$

When $q>0$ (or $q\ge 1$, $p=1$) we also have

$$
\begin{aligned}
w(p,q,1) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^p m^q (n+m)} \\
&= \sum_{k=1}^{p-1}(-1)^{k-1}\zeta(p-k+1)\zeta(q+k)+(-1)^{p-1}S(1,p+q).
\end{aligned}
\tag{2.13}
$$

3. Remarks. Our notation $S(p,q)$ corresponds to $S_{p,q}$ in [5]. The authors of [2] use the sums $\zeta(p,q)$, which equal $S(q,p)-\zeta(p+q)$.

The representation (2.11) has strong and weak points. One good feature is that q need not be an integer. A weak point is that the right-hand side in (2.11) is not explicitly symmetrical in p and q, while obviously $w(p,q,r)=w(q,p,r)$. Moreover, the right-hand side has too many terms. For instance, when $q=0$ (2.11) becomes

$$
\begin{aligned}
w(p,0,r) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^p (n+m)^r} \\
&= (-1)^{p-1}S(r,p)+\sum_{k=0}^{p-2}(-1)^k\binom{r+k-1}{r-1}\zeta(p-k)\zeta(r+k) \\
&\quad + \frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-2}\frac{(r+p-k-2)!}{(r-k-1)!}\left[S(k+1,r+p-k-1)-\zeta(k+1)\zeta(r+p-k-1)\right]
\end{aligned}
\tag{3.1}
$$

(here the term $(-1)^{p-1}S(r,p)$ is written separately on purpose).

At the same time

$$
w(p,0,r)=\sum_{n=1}^{\infty}\frac{1}{n^p}\sum_{m=1}^{\infty}\frac{1}{(n+m)^r}=\sum_{n=1}^{\infty}\frac{1}{n^p}\left(\zeta(r)-H_n^{(r)}\right)=\zeta(p)\zeta(r)-S(r,p) \tag{3.2}
$$

which is much shorter. However, we can benefit from this situation if we compare the two representations of $w(p,0,r)$ and derive relations for the single and double Euler sums. For instance, when p is odd, we can solve for $S(r,p)$ to get

$$
\begin{aligned}
2S(r,p) &= \sum_{k=1}^{p-2}(-1)^{k+1}\binom{r+k-1}{r-1}\zeta(p-k)\zeta(r+k) \\
&\quad + \frac{(-1)^{p-1}}{(p-1)!}\sum_{k=0}^{r-2}\frac{(r+p-k-2)!}{(r-k-1)!}\left[S(k+1,r+p-k-1)-\zeta(k+1)\zeta(r+p-k-1)\right],
\end{aligned}
\tag{3.3}
$$

that is, $S(r,p)$ can be expressed in terms of single zeta values and $S(k,l)$, with $k<r$, $k+l=r+p$.

4. Other sums. It is interesting to consider also the sum

$$
u(p,q,r)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^p m^q (n^r+m^r)} \tag{4.1}
$$

and compare it to $w(p,q,r)$. Here one can write

$$u(p,q,r)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n^r+m^r-n^r}{n^{p+r}m^q(n^r+m^r)}=\zeta(p+r)\zeta(q)-u(p+r,q-r,r). \tag{4.2}$$

Let $q>p$. We observe that if $(q-p)/r$ is odd, repeating this step $(q-p)/r$ times, we get

$$u(p,q,r)=\sum_{j=1}^{(q-p)/r}(-1)^{j-1}\zeta(p+jr)\zeta(q-(j-1)r)-u(q,p,r) \tag{4.3}$$

from where, because of the symmetry $u(p,q,r)=u(q,p,r)$, we obtain the following proposition.

Proposition 4.1. *For all $q>p\ge 1$, $r\ge 1$ with $(q-p)/r$ odd,*

$$u(p,q,r)=\frac{1}{2}\sum_{j=1}^{(q-p)/r}(-1)^{j-1}\zeta(p+jr)\zeta(q-(j-1)r). \tag{4.4}$$

Note that p,q,r need not be integers. The only restrictions are those listed above. When $r=1$ we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^pm^q(n+m)}=\frac{1}{2}\sum_{j=1}^{q-p}(-1)^{j-1}\zeta(p+j)\zeta(q-j+1) \tag{4.5}$$

which can be compared to (2.13). This gives the well-known expression of $S(1,p+q)$ in terms of zeta values. To make this more explicit we set $p=1$ and $q\ge 2$. Then from (2.13),

$$w(1,q-1,1)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{nm^{q-1}(n+m)}=S(1,q). \tag{4.6}$$

This is the same as $u(1,q-1,1)$. When q is odd we find from (4.5) (with $p=1$ and $q-1$ in the place of q)

$$S(1,q)=\frac{1}{2}\sum_{j=1}^{q-2}(-1)^{j-1}\zeta(j+1)\zeta(q-j) \tag{4.7}$$

which is a variant of Euler's formula for the sum $S(1,q)$ (see [5, Theorem 2.2]).

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