

MULTIVARIABLE DIMENSION POLYNOMIALS AND NEW INVARIANTS OF DIFFERENTIAL FIELD EXTENSIONS

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ABSTRACT. We introduce a special type of reduction in the ring of differential polynomials and develop the appropriate technique of characteristic sets that allows to generalize the classical Kolchin's theorem on differential dimension polynomial and find new differential birational invariants of a finitely generated differential field extension.

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1. Introduction. The role of Hilbert polynomials in commutative algebra and algebraic geometry is well known. A similar role in differential algebra is played by differential dimension polynomials. The notion of a differential dimension polynomial was introduced by Kolchin in [6], but the problems and ideas that had led to this concept have essentially more long history. Actually, the differential dimension polynomial describes in exact terms the freedom degree of a dynamic system as well as the number of arbitrary constants in the general solution of a system of algebraic differential equations. The first attempts of such a description were made in the 19th century by Jacobi [3] who estimated the number of algebraically independent constants in the general solution of a system of ordinary linear differential equations. Later on, Jacobi's results were extended to some nonlinear systems, but in the general case the problem of such estimation (known as the problem of Jacobi's bound) remains open.

Differential algebra as a separate area of mathematics is largely due to its founder Ritt (1893–1951) and Kolchin (1916–1991). In 1964 Kolchin proved his famous theorem on differential dimension polynomial (see [Theorem 2.1](#) below) that lies in the foundation of the theory of differential dimension. At the International Congress of Mathematicians in Moscow (1966) Kolchin formulated the main problems and outlined the most perspective directions of research connected with the differential dimension polynomial. Later on the results obtained in this area were included into his famous monograph [7] that hitherto remains the most fundamental work on differential algebra.

Discussing the history of creation of the differential dimension theory, one should note that in 1953 Einstein [2] introduced a concept of strength of a system of differential equations as a certain function of integer argument associated with the system. In 1980 Mikhalëv and Pankrat'ev [12] showed that this function actually coincides with the appropriate differential dimension polynomial and found the strength of some well-known systems of partial differential equations using methods of differential algebra.

The intensive study of Kolchin's differential dimension polynomials began at the end of the sixties with the series of works by Johnson [4, 5, 15] who developed the technique of dimension polynomials for differential modules and applied it to the study of some classical problems of differential algebra. In particular, he characterized the Krull dimension of finitely generated differential algebras, developed the theory of local differential algebras, and proved a special case of Janet conjecture. A number of interesting properties and applications of differential dimension polynomials were found by Kondrat'eva, Levin, Mikhal'ev, Pankrat'ev, Sit, and some other mathematicians (see [9, 10, 11, 12, 13, 14]). One of the most important directions of this study was the search for new differential birational invariants connected with the differential dimension polynomials. Here we should mention the results of Sit [13] who showed that the set of all differential dimension polynomials is well ordered with respect to some natural ordering and introduced the notion of the minimal differential dimension polynomial associated with a differential field extension.

In this paper, we introduce a special type of reduction in a ring of differential polynomials over a differential field of zero characteristic whose basic set is represented as a disjoint union of its subsets. Using the idea of the Gröbner basis method introduced in [1], we develop the appropriate technique of characteristic sets that allows to prove the existence and outline a method of computation of multivariable dimension polynomials associated with a finitely generated differential field extension. In particular, we obtain a generalization of the Kolchin's theorem and find new differential birational invariants.

2. Preliminaries. Throughout the paper \mathbb{Z} , \mathbb{N} , and \mathbb{Q} denote the sets of all integers, all nonnegative integers, and all rational numbers, respectively. By a ring we always mean an associative ring with a unit. Every ring homomorphism is unitary (maps unit onto unit), every subring of a ring contains the unit of the ring, and every algebra over a commutative ring is unitary. Unless otherwise indicated, every field is supposed to have zero characteristic.

A *differential ring* is a commutative ring R considered together with a finite set Δ of mutually commuting derivations of the ring R into itself. The set Δ is called a *basic set* of the differential ring R that is also called a Δ -ring. A subring (ideal) R_0 of a Δ -ring R is called a *differential* or Δ -subring of R (respectively, *differential* or Δ -ideal of R) if R_0 is closed with respect to the action of any operator $\delta \in \Delta$. If a differential (Δ -)ring is a field, it is called a *differential* (or Δ -)field.

Let R and S be two differential rings with the same basic set $\Delta = \{\delta_1, \dots, \delta_m\}$, so that elements of the set Δ act on each of the rings as mutually commuting derivations. A ring homomorphism $\phi : R \rightarrow S$ is called a *differential* or Δ -homomorphism if $\phi(\delta a) = \delta\phi(a)$ for any $\delta \in \Delta$, $a \in R$.

In what follows, K denotes a differential field whose basic set of derivation operators Δ is a union of p disjoint finite sets ($p \geq 1$) : $\Delta = \Delta_1 \cup \dots \cup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($i = 1, \dots, p$ and m_1, \dots, m_p are positive integers whose sum is equal to m , the number of elements of the set Δ). In other words, we fix a partition of the basic set Δ . Let Θ_i be the free commutative semigroup generated by the elements of the set Δ_i ($i = 1, \dots, p$) and Θ the free commutative semigroup generated

by the whole set Δ . For any element $\theta = \delta_{11}^{k_{11}} \cdots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \cdots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, the numbers $\text{ord}_i \theta = \sum_{j=1}^{m_i} k_{ij}$ ($i = 1, \dots, p$) and $\text{ord } \theta = \sum_{i=1}^p \text{ord}_i \theta$ will be called the order of θ with respect to Δ_i and the order of θ , respectively. As usual, if $\theta, \theta' \in \Theta$, we say that θ' divides θ if $\theta = \theta'' \theta'$ for some element $\theta'' \in \Theta$. By the least common multiple of the elements $\theta_1 = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ij}^1}, \dots, \theta_q = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ij}^q} \in \Theta$ we mean the element $\theta = \prod_{i=1}^p \prod_{j=1}^{m_i} \delta_{ij}^{k_{ij}}$, where $k_{ij} = \max\{k_{ijl} \mid 1 \leq l \leq q\}$ ($1 \leq i \leq p, 1 \leq j \leq m_i$). This element will be denoted by $\text{lcm}(\theta_1, \dots, \theta_q)$.

For any $r_1, \dots, r_p, r \in \mathbb{N}$, the sets $\{\theta \in \Theta \mid \text{ord}_i \theta \leq r_i \text{ for } i = 1, \dots, p\}$ and $\{\theta \in \Theta \mid \text{ord } \theta \leq r\}$ will be denoted by $\Theta(r_1, \dots, r_p)$ and $\Theta(r)$, respectively. Furthermore, we set $T\xi = \{\theta(\xi) \mid \theta \in T\}$ for any $\xi \in K, T \subseteq \Theta$.

Below we will consider p orderings $<_1, \dots, <_p$ on Θ defined as follows:

$$\theta = \delta_{11}^{k_{11}} \cdots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \cdots \delta_{pm_p}^{k_{pm_p}} <_i \theta' = \delta_{11}^{l_{11}} \cdots \delta_{1m_1}^{l_{1m_1}} \delta_{21}^{l_{21}} \cdots \delta_{pm_p}^{l_{pm_p}} \tag{2.1}$$

if and only if the vector $(\text{ord}_i \theta, \text{ord } \theta, \text{ord}_1 \theta, \dots, \text{ord}_{i-1} \theta, \text{ord}_{i+1} \theta, \dots, \text{ord}_p \theta, k_{i1}, \dots, k_{im}, k_{11}, \dots, k_{1m_1}, k_{21}, \dots, k_{i-1, m_{i-1}}, k_{i+1, 1}, \dots, k_{pm_p})$ is less than the vector $(\text{ord}_i \theta', \text{ord } \theta', \text{ord}_1 \theta', \dots, \text{ord}_{i-1} \theta', \text{ord}_{i+1} \theta', \dots, \text{ord}_p \theta', l_{i1}, \dots, l_{im}, l_{11}, \dots, l_{1m_1}, l_{21}, \dots, l_{i-1, m_{i-1}}, l_{i+1, 1}, \dots, l_{pm_p})$ with respect to the lexicographic order on \mathbb{N}^{m+p+1} .

If R is a differential ring with a basic set Δ and $\Sigma \subseteq R$, then the intersection of all Δ -ideals of R containing the set Σ is, obviously, the smallest Δ -ideal of R containing Σ . This ideal is denoted by $[\Sigma]$. (It is clear that $[\Sigma]$ is generated, as an ideal, by the set $\{\Theta\xi \mid \xi \in \Sigma\}$.) If the set Σ is finite, $\Sigma = \{\xi_1, \dots, \xi_q\}$, we say that the Δ -ideal $I = [\Sigma]$ is finitely generated (we write this as $I = [\xi_1, \dots, \xi_q]$) and call ξ_1, \dots, ξ_q differential or Δ -generators of I .

A subfield K_0 of the Δ -field K is said to be a differential (or Δ -) subfield of K if $\delta(K_0) \subseteq K_0$ for any $\delta \in \Delta$. If K_0 is a Δ -subfield of the Δ -field K and $\Sigma \subseteq K$, then the intersection of all Δ -subfields of K containing K_0 and Σ is the unique Δ -subfield of K containing K_0 and Σ and contained in every Δ -subfield of K containing K_0 and Σ . It is denoted by $K_0\langle \Sigma \rangle$. If $K = K_0\langle \Sigma \rangle$ and the set Σ is finite, $\Sigma = \{\eta_1, \dots, \eta_n\}$, then K is said to be a finitely generated Δ -extension of K_0 with the set of Δ -generators $\{\eta_1, \dots, \eta_n\}$. In this case we write $K = K_0\langle \eta_1, \dots, \eta_n \rangle$. It is easy to see that the field $K_0\langle \eta_1, \dots, \eta_n \rangle$ coincides with the field $K_0\langle \theta\eta_i \mid \theta \in \Theta, 1 \leq i \leq n \rangle$.

Now we can formulate the Kolchin's theorem on differential dimension polynomial (see [7, Chapter 2, Theorem 6]). As usual, $\binom{t}{k}$ ($k \in \mathbb{Z}, k \geq 1$) denotes the polynomial $t(t-1) \cdots (t-k+1)/k!$ in one variable t , $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if $k < 0$.

THEOREM 2.1. *Let K be a differential field with a basic set $\Delta = \{\delta_1, \dots, \delta_m\}$ and L a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a polynomial $\omega_{\eta|K}(t)$ in one variable t with rational coefficients (called a differential dimension polynomial of the extension) such that*

- (i) $\omega_{\eta|K}(r) = \text{trdeg}_K K\langle \theta\eta_j \mid \theta \in \Theta(r), 1 \leq j \leq n \rangle$ for all sufficiently large integers r ;
- (ii) $\text{deg } \omega_{\eta|K} \leq m$ and the polynomial $\omega_{\eta|K}(t)$ can be written as $\omega_{\eta|K}(t) = \sum_{i=1}^m a_i \binom{t+i}{i}$, where a_0, \dots, a_m are some integers;

- (iii) the degree d of the polynomial $\omega_{\eta|K}$ and the coefficients a_m and a_d do not depend on the choice of the system of Δ -generators η of the extension L/K (clearly, $a_d \neq a_m$ if and only if $d < m$, that is, $a_m = 0$). In other words, d , a_m , and a_d are differential birational invariants of the extension. Moreover, the coefficient a_m is equal to the differential transcendence degree of L over K , that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\theta \xi_i \mid \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K .

If $Y = \{y_1, \dots, y_n\}$ is a finite set of symbols, then one can consider the countable set of symbols $\Theta Y = \{\theta y_j \mid \theta \in \Theta, 1 \leq j \leq n\}$ (called *terms*) and the polynomial ring $R = K[\{\theta y_j \mid \theta \in \Theta, 1 \leq j \leq n\}]$ in the set of indeterminates ΘY over the differential field K . This polynomial ring is naturally viewed as a Δ -ring where $\delta(\theta y_j) = (\delta\theta)y_j$ ($\delta \in \Delta, \theta \in \Theta, 1 \leq j \leq n$) and the elements of Δ act on the coefficients of the polynomials of R as they act in the field K . The ring R is called the *ring of differential* (or Δ -) *polynomials* in the set of differential (Δ -)indeterminates y_1, \dots, y_n over the Δ -field K . This ring is denoted by $K\{y_1, \dots, y_n\}$ and its elements are called differential (or Δ -) polynomials.

The set of all terms ΘY will be considered together with p orderings that correspond to the orderings of the semigroup Θ and that are denoted by the same symbols $<_1, \dots, <_p$. These orderings of ΘY are defined as follows: $\theta y_j <_i \theta' y_k$ ($\theta, \theta' \in \Theta, 1 \leq j, k \leq n, 1 \leq i \leq p$) if and only if $\theta <_i \theta'$ or $\theta = \theta'$ and $j < k$.

By the i th order of a term $u = \theta y_j$ we mean the number $\text{ord}_i u = \text{ord}_i \theta$. The number $\text{ord } u = \text{ord } \theta$ is called the order of the term u .

We say that a term $u = \theta y_j$ is divisible by a term $v = \theta' y_j$ (or u is a *multiple* of v) and write $v \mid u$, if $i = j$ and $\theta' \mid \theta$. For any terms $u_1 = \theta_1 y_j, \dots, u_q = \theta_q y_j$ containing the same Δ -indeterminate y_j ($1 \leq j \leq n$), the term $\text{lcm}(\theta_1, \dots, \theta_q) y_j$ is called the least common multiple of u_1, \dots, u_q , it is denoted by $\text{lcm}(u_1, \dots, u_q)$.

If $A \in K\{y_1, \dots, y_n\}$, $A \notin K$, and $1 \leq i \leq p$, then the highest with respect to the ordering $<_i$ term that appears in A is called the i -leader of the Δ -polynomial A . It is denoted by $u_A^{(i)}$. If A is written as a polynomial in one variable $u_A^{(1)}$, $A = I_d(u_A^{(1)})^d + I_{d-1}(u_A^{(1)})^{d-1} + \dots + I_0$ (the Δ -polynomials I_d, I_{d-1}, \dots, I_0 do not contain $u_A^{(1)}$), then I_d is called the *leading coefficient* of the Δ -polynomial A and the partial derivative $\partial A / \partial u_A^{(1)} = dI_d(u_A^{(1)})^{d-1} + (d-1)I_{d-1}(u_A^{(1)})^{d-2} + \dots + I_1$ is called the *separant* of A . The leading coefficient and the separant of a Δ -polynomial A are denoted by I_A and S_A , respectively.

DEFINITION 2.2. Let A and B be two Δ -polynomials from $K\{y_1, \dots, y_n\}$. We say that A has a lower rank than B and write $rk A < rk B$ if either $A \in K, B \notin K$, or the vector $(u_A^{(1)}, \text{deg}_{u_A^{(1)}} A, \text{ord}_2 u_A^{(2)}, \dots, \text{ord}_p u_A^{(p)})$ is less than the vector $(u_B^{(1)}, \text{deg}_{u_B^{(1)}} B, \text{ord}_2 u_B^{(2)}, \dots, \text{ord}_p u_B^{(p)})$ with respect to the lexicographic order (where $u_A^{(1)}$ and $u_B^{(1)}$ are compared with respect to $<_1$ and all other coordinates of the vectors are compared with respect to the natural order on \mathbb{N}). If the two vectors are equal (or $A \in K$ and $B \in K$) we say that the Δ -polynomials A and B are of the same rank and write $rk A = rk B$.

Let K be a Δ -field and $G = K\langle \eta_1, \dots, \eta_n \rangle$ a finitely generated Δ -extension of K with a set of generators $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a natural Δ -homomorphism Φ_η from the ring of Δ -polynomials $K\{y_1, \dots, y_n\}$ to G such that $\Phi_\eta(a) = a$ for any

$a \in K$ and $\Phi_\eta(y_j) = \eta_j$ for $j = 1, \dots, n$. If $A \in K\{y_1, \dots, y_n\}$, then the element $\Phi_\eta(A)$ is called the *value* of the Δ -polynomial A at η , it is denoted by $A(\eta)$. Obviously, the kernel P of the mapping Φ_η is a prime Δ -ideal of the ring $K\{y_1, \dots, y_n\}$. This ideal is called the *defining ideal* of η over K or the defining ideal of the Δ -field extension $G = K\langle\eta_1, \dots, \eta_n\rangle$. It is easy to see that if the quotient field Q of the factor ring $\bar{R} = K\{y_1, \dots, y_n\}/P$ is considered as a Δ -field (where $\delta(u/v) = (v\delta(u) - u\delta(v))/v^2$ for any $u, v \in \bar{R}$), then Q is naturally Δ -isomorphic to the field G . (The appropriate Δ -isomorphism is identical on K and maps the images of the Δ -indeterminates y_1, \dots, y_n in the factor ring \bar{R} onto the elements η_1, \dots, η_n , respectively.)

3. Numerical polynomials. A polynomial $f(t_1, \dots, t_p)$ in p variables t_1, \dots, t_p ($p \in \mathbb{N}$, $p \geq 1$) with rational coefficients is called *numerical* if $f(t_1, \dots, t_p) \in \mathbb{Z}$ for all sufficiently large $(t_1, \dots, t_p) \in \mathbb{Z}^p$, that is, there exist $s_1, \dots, s_p \in \mathbb{Z}$ such that $f(r_1, \dots, r_p) \in \mathbb{Z}$ as soon as $(r_1, \dots, r_p) \in \mathbb{Z}^p$ and $r_i \geq s_i$ for all $i = 1, \dots, p$.

It is clear that any polynomial with integer coefficients is numerical. As an example of a numerical polynomial with noninteger coefficients one can consider a polynomial of the form $\prod_{i=1}^p \binom{t_i}{m_i}$ where $m_1, \dots, m_p \in \mathbb{N}$ ($p \in \mathbb{N}$, $p \geq 1$).

If $f(t_1, \dots, t_p)$ is a numerical polynomial, then $\deg f$ and $\deg_{t_i} f$ ($1 \leq i \leq p$) will denote the total degree of f and the degree of f relative to the variable t_i , respectively. The following theorem proved in [8] gives the “canonical” representation of a numerical polynomial in several variables.

THEOREM 3.1. *Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables t_1, \dots, t_p , and let $\deg_{t_i} f = m_i$ ($m_1, \dots, m_p \in \mathbb{N}$). Then the polynomial $f(t_1, \dots, t_p)$ can be represented in the form*

$$f(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p} \tag{3.1}$$

with integer coefficients $a_{i_1 \dots i_p}$ ($0 \leq i_k \leq m_k$ for $k = 1, \dots, p$) that are uniquely defined by the numerical polynomial.

In the rest of the section we deal with subsets of \mathbb{N}^m where the positive integer m is represented as a sum of p nonnegative integers m_1, \dots, m_p ($p \in \mathbb{N}$, $p \geq 1$). In other words, we fix a partition (m_1, \dots, m_p) of the number m .

If $\mathcal{A} \subseteq \mathbb{N}^m$ and $r_1, \dots, r_p \in \mathbb{N}$, then $\mathcal{A}(r_1, \dots, r_p)$ will denote the set $\{(a_1, \dots, a_m) \in \mathcal{A} \mid a_1 + \cdots + a_{m_1} \leq r_1, a_{m_1+1} + \cdots + a_{m_1+m_2} \leq r_2, \dots, a_{m_1+\cdots+m_{p-1}+1} + \cdots + a_m \leq r_p\}$. Furthermore, $V_{\mathcal{A}}$ will denote the set of all m -tuples $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ that are not greater than or equal to any m -tuple from \mathcal{A} with respect to the product order on \mathbb{N}^m . (Recall that the product order on the set \mathbb{N}^k ($k \in \mathbb{N}, k \geq 1$) is a partial order \leq_p such that $c = (c_1, \dots, c_k) \leq_p c' = (c'_1, \dots, c'_k)$ if and only if $c_i \leq c'_i$ for all $i = 1, \dots, k$. If $c \leq_p c'$ and $c \neq c'$, we write $c <_p c'$). Clearly, an element $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ belongs to $V_{\mathcal{A}}$ if and only if for any $(a_1, \dots, a_m) \in \mathcal{A}$, there exists $i \in \mathbb{N}$, $1 \leq i \leq m$, such that $a_i > v_i$.

The following two theorems proved in [8] generalize the well-known Kolchin’s result on the numerical polynomials associated with subsets of \mathbb{N} (see [7, Chapter 0, Lemma 17]) and give the explicit formula for the numerical polynomials in p variables associated with a finite subset of \mathbb{N}^m .

THEOREM 3.2. *Let \mathcal{A} be a subset of \mathbb{N}^m where $m = m_1 + \dots + m_p$ for some nonnegative integers m_1, \dots, m_p ($p \geq 1$). Then there exists a numerical polynomial $\omega_{\mathcal{A}}(t_1, \dots, t_p)$ with the following properties:*

- (i) $\omega_{\mathcal{A}}(r_1, \dots, r_p) = \text{Card}V_{\mathcal{A}}(r_1, \dots, r_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$ (as usual, $\text{Card}M$ denotes the number of elements of a finite set M).
- (ii) $\deg \omega_{\mathcal{A}} \leq m$ and $\deg_{t_i} \omega_{\mathcal{A}} \leq m_i$ for $i = 1, \dots, p$.
- (iii) $\deg \omega_{\mathcal{A}} = m$ if and only if $\mathcal{A} = \emptyset$. In this case, $\omega_{\mathcal{A}}(t_1, \dots, t_p) = \prod_{i=1}^p \binom{t_i + m_i}{m_i}$.
- (iv) $\omega_{\mathcal{A}}$ is a zero polynomial if and only if $(0, \dots, 0) \in \mathcal{A}$.

The polynomial $\omega_{\mathcal{A}}(t_1, \dots, t_p)$ whose existence is stated by [Theorem 3.2](#) is called the *dimension polynomial of the set $\mathcal{A} \subseteq \mathbb{N}^m$ associated with the partition (m_1, \dots, m_p) of m* . If $p = 1$, the polynomial $\omega_{\mathcal{A}}$ is called the *Kolchin polynomial* of the set \mathcal{A} .

THEOREM 3.3. *Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a finite subset of \mathbb{N}^m and (m_1, \dots, m_p) ($p \geq 1$) a partition of m . Let $a_i = (a_{i1}, \dots, a_{im})$ ($1 \leq i \leq n$) and for any $l \in \mathbb{N}$, $0 \leq l \leq n$, let $\Gamma(l, n)$ denote the set of all l -element subsets of the set $\mathbb{N}_n = \{1, \dots, n\}$. Furthermore, for any $\sigma \in \Gamma(l, p)$, let $\bar{a}_{\sigma j} = \max\{a_{ij} \mid i \in \sigma\}$ ($1 \leq j \leq m$) and $b_{\sigma j} = \sum_{h \in \sigma_j} \bar{a}_{\sigma h}$. Then*

$$\omega_{\mathcal{A}}(t_1, \dots, t_p) = \sum_{l=0}^n (-1)^l \sum_{\sigma \in \Gamma(l, p)} \prod_{j=1}^n \binom{t_j + m_j - b_{\sigma j}}{m_j}. \tag{3.2}$$

It is clear that if $\mathcal{A} \subseteq \mathbb{N}^m$ and \mathcal{A}' is the set of all minimal elements of the set \mathcal{A} with respect to the product order on \mathbb{N}^m , then the set \mathcal{A}' is finite and $\omega_{\mathcal{A}}(t_1, \dots, t_p) = \omega_{\mathcal{A}'}(t_1, \dots, t_p)$. Thus, [Theorem 3.3](#) gives an algorithm that allows to find a numerical polynomial associated with any subset of \mathbb{N}^m (and with a given partition of m): one should first find the set of all minimal points of the subset and then apply [Theorem 3.3](#).

4. Reduction in the ring of differential polynomials. In what follows we keep the notation and conventions of [Section 2](#). In particular, $K\{y_1, \dots, y_n\}$ denotes the ring of Δ -polynomials over a differential field K whose basic set Δ is a union of p disjoint sets: $\Delta = \Delta_1 \cup \dots \cup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($1 \leq i \leq p$).

DEFINITION 4.1. Let $A, B \in K\{y_1, \dots, y_n\}$ and $A \notin K$. The Δ -polynomial B is said to be reduced with respect to A if the following two conditions hold:

- (i) B does not contain any term $\theta u_A^{(1)}$ ($\theta \in \Theta, \theta \neq 1$) such that $\text{ord}_i(\theta u_A^{(i)}) \leq \text{ord}_i u_B^{(i)}$ for $i = 2, \dots, p$.
- (ii) If B contains $u_A^{(1)}$, then either $\text{ord}_j u_B^{(j)} < \text{ord}_j u_A^{(j)}$ for some j ($2 \leq j \leq p$) or $\text{ord}_i u_A^{(i)} \leq \text{ord}_i u_B^{(i)}$ for all $i = 2, \dots, p$ and $\deg_{u_A^{(1)}} B < \deg_{u_A^{(1)}} A$.

A Δ -polynomial B is said to be reduced with respect to a set of Δ -polynomials $\Sigma \subseteq K\{y_1, \dots, y_n\}$ if B is reduced with respect to every element of Σ .

DEFINITION 4.2. A set of Δ -polynomials $\Sigma \subseteq K\{y_1, \dots, y_n\}$ is called *autoreduced* if $\Sigma \cap K = \emptyset$ and every element of Σ is reduced with respect to any other element of this set.

The proof of the following lemma can be found in [[7](#), Chapter 0, Section 17].

LEMMA 4.3. *Let $\mathbb{N}_n = \{1, \dots, n\}$ and let A be an infinite subset of $\mathbb{N}^m \times \mathbb{N}_n$ ($m, n \in \mathbb{N}$, $n \geq 1$). Then there exists an infinite sequence of elements of A , strictly increasing relative to the product order, in which every element has the same projection on \mathbb{N}_n .*

This result implies the following statement that will be used below.

LEMMA 4.4. *Let S be an infinite set of terms in the ring $K\{y_1, \dots, y_n\}$. Then there exists an index j ($1 \leq j \leq n$) and an infinite sequence of terms $\theta_1 y_j, \theta_2 y_j, \dots, \theta_k y_j, \dots \in S$ such that $\theta_k \mid \theta_{k+1}$ for all $k = 1, 2, \dots$.*

THEOREM 4.5. *Every autoreduced set of Δ -polynomials is finite.*

PROOF. Suppose that Σ is an infinite autoreduced subset of $K\{y_1, \dots, y_n\}$. Then Σ contains an infinite set Σ' such that all Δ -polynomials from Σ' have different 1-leaders. Indeed, if it is not so, then there exists an infinite set $\Sigma_1 \subseteq \Sigma$ such that all Δ -polynomials from Σ_1 have the same 1-leader u . By Lemma 4.3, the infinite set $\{(\text{ord}_2 u_A^{(2)}, \dots, \text{ord}_p u_A^{(p)}) \mid A \in \Sigma_1\}$ contains a nondecreasing infinite sequence $(\text{ord}_2 u_{A_1}^{(2)}, \dots, \text{ord}_p u_{A_1}^{(p)}) \leq_p (\text{ord}_2 u_{A_2}^{(2)}, \dots, \text{ord}_p u_{A_2}^{(p)}) \leq_p \dots$ ($A_1, A_2, \dots \in \Sigma_1$ and \leq_p denotes the product order on \mathbb{N}^{p-1}). Since the sequence $\{\deg_u A_i \mid i = 1, 2, \dots\}$ cannot strictly decrease, there exists two indices i and j such that $i < j$ and $\deg_u A_i \leq \deg_u A_j$. We obtain that A_j is reduced with respect to A_i that contradicts the fact that Σ is an autoreduced set.

Thus, we can assume that elements of Σ have distinct 1-leaders. By Lemma 4.4, there exists an infinite sequence B_1, B_2, \dots of elements of Σ such that $u_{B_i}^{(1)} \mid u_{B_{i+1}}^{(1)}$ for $i = 1, 2, \dots$. Let $k_{ij} = \text{ord}_j u_{B_i}^{(1)}$ and $l_{ij} = \text{ord}_j u_{B_i}^{(j)}$ ($2 \leq j \leq p$). Obviously, $l_{ij} \geq k_{ij}$ ($i = 1, 2, \dots; j = 2, \dots, p$), so that $\{(l_{i2} - k_{i2}, \dots, l_{ip} - k_{ip}) \mid i = 1, 2, \dots\} \subseteq \mathbb{N}^{p-1}$. By Lemma 4.3, there exists an infinite sequence of indices $i_1 < i_2 < \dots$ such that $(l_{i_1 2} - k_{i_1 2}, \dots, l_{i_1 p} - k_{i_1 p}) \leq_p (l_{i_2 2} - k_{i_2 2}, \dots, l_{i_2 p} - k_{i_2 p}) \leq_p \dots$. Then for any $j = 2, \dots, p$, we have

$$\text{ord}_j \left(\frac{u_{B_{i_2}}^{(1)}}{u_{B_{i_1}}^{(1)}} u_{B_{i_1}}^{(j)} \right) = k_{i_2 j} - k_{i_1 j} + l_{i_1 j} \leq k_{i_2 j} + l_{i_2 j} - k_{i_2 j} = l_{i_2 j} = \text{ord}_j u_{B_{i_2}}^{(j)}, \quad (4.1)$$

so that B_{i_2} contains a term $\theta u_{B_{i_1}}^{(1)} = u_{B_{i_2}}^{(1)}$ such that $\theta \neq 1$ and $\text{ord}_j(\theta u_{B_{i_1}}^{(j)}) \leq \text{ord}_j u_{B_{i_2}}^{(j)}$ for $j = 2, \dots, p$. Thus, the Δ -polynomial B_{i_2} is reduced with respect to B_{i_1} that contradicts the fact that Σ is an autoreduced set. This completes the proof of the theorem. □

Below, the elements of an autoreduced set will be always arranged in order of increasing rank. (Therefore, if we consider an autoreduced set of Δ -polynomials $\Sigma = \{A_1, \dots, A_r\}$, then $rk A_1 < \dots < rk A_r$.)

The following two theorems can be proven precisely in the same way as their classical analogs (see [7, Chapter 1, Corollary to Lemma 6 and Proposition 3, page 78]).

THEOREM 4.6. *Let $\Sigma = \{A_1, \dots, A_r\}$ be an autoreduced set in the ring $K\{y_1, \dots, y_n\}$ and let B be a Δ -polynomial. Then there exist a Δ -polynomial B_0 and nonnegative integers p_i, q_i ($1 \leq i \leq r$) such that B_0 is reduced with respect to Σ , $rk B_0 \leq rk B$, and $\prod_{i=1}^r I_{A_i}^{p_i} S_{A_i}^{q_i} B \equiv B_0 \pmod{[\Sigma]}$.*

DEFINITION 4.7. Let $\Sigma = \{A_1, \dots, A_r\}$ and $\Sigma' = \{B_1, \dots, B_s\}$ be two autoreduced sets in the ring of differential polynomials $K\{y_1, \dots, y_n\}$. An autoreduced set Σ is said to have lower rank than Σ' if one of the following two cases holds:

- (1) there exists $k \in \mathbb{N}$ such that $k \leq \min\{r, s\}$, $rkA_i = rkB_i$ for $i = 1, \dots, k-1$ and $rkA_k < rkB_k$;
 - (2) $r > s$ and $rkA_i = rkB_i$ for $i = 1, \dots, s$.
- If $r = s$ and $rkA_i = rkB_i$ for $i = 1, \dots, r$, then Σ is said to have the same rank as Σ' .

THEOREM 4.8. *In every nonempty family of autoreduced sets of differential polynomials there exists an autoreduced set of lowest rank.*

Let J be an ideal of the ring $K\{y_1, \dots, y_n\}$. Since the family of all autoreduced subsets of J is not empty (e.g., it contains the empty set), [Theorem 4.8](#) shows that the ideal J contains an autoreduced subset of lowest rank.

DEFINITION 4.9. Let J be an ideal of the ring of differential polynomials $K\{y_1, \dots, y_n\}$. Then an autoreduced subset of J of lowest rank is called a characteristic set of the ideal J .

THEOREM 4.10. *Let $\Sigma = \{A_1, \dots, A_d\}$ be a characteristic set of an ideal J of the ring of Δ -polynomials $R = K\{y_1, \dots, y_n\}$. Then an element $B \in R$ is reduced with respect to the set Σ if and only if $B = 0$.*

PROOF. Suppose that $B \neq 0$. Then B and elements of Σ whose rank is lower than the rank of B form an autoreduced set Σ' . It is easy to see that Σ' has a lower rank than Σ that contradicts the fact that Σ is a characteristic set of the ideal J . □

THEOREM 4.11. *Let J be a cyclic differential ideal of the ring of Δ -polynomials $R = K\{y_1, \dots, y_n\}$ generated by a linear Δ -polynomial f . Then $\{f\}$ is a characteristic set of the Δ -ideal $J = [f]$.*

PROOF. First of all, we show that no nonzero element of J is reduced with respect to f . Let $0 \neq h \in J$, and let k be the smallest positive integer such that h can be written as

$$h = g_1\theta_1f + \dots + g_k\theta_kf, \tag{4.2}$$

for some pairwise distinct elements $\theta_1, \dots, \theta_k \in \Theta$ and some $g_1, \dots, g_k \in R$. In what follows, we suppose that $k > 1$ (clearly, an element of the form $g\theta f$ ($g \in R, \theta \in \Theta$) is not reduced with respect to f) and $\theta_1 <_1 \dots <_1 \theta_k$. Furthermore, it is obvious that $\theta_k f \nmid g_j$ for $j = 1, \dots, k-1$ (otherwise, h is a linear combination of $k-1$ elements from Θf).

Since $\theta_k f$ is linear with respect to $\theta_k u_f^{(1)}$, one can write each Δ -polynomial g_j ($1 \leq j \leq k-1$) as $g_j = g'_j + g''_j(\theta_k f)$, where $g'_j, g''_j \in R$ and g'_j does not contain $\theta_k u_f^{(1)}$. Then $h = g'_1\theta_1f + \dots + g'_{k-1}\theta_{k-1}f + g\theta_kf$ for some $g \in R$.

Since $g\theta_kf$ contains $\theta_k u_f^{(1)}$ and none of $g'_j\theta_jf$ ($1 \leq j \leq k-1$) contains this term, the Δ -polynomial h contains $\theta_k u_f^{(1)}$ and $rkf \leq rk h$. Similarly, if θ_{j_i} is the maximal element of the set $\{\theta_1, \dots, \theta_k\}$ relative to the order $<_i$ on Θ ($2 \leq i \leq p$), then h contains $\theta_{j_i} u_f^{(i)}$, so that $\text{ord}_i(\theta_k u_f^{(i)}) \leq \text{ord}_i(\theta_{j_i} u_f^{(i)}) \leq \text{ord}_i u_h^{(i)}$. It follows that h is reduced with

respect to f and f is an element of the lowest rank in J . Therefore, if $\Sigma = \{h_1, \dots, h_l\}$ is a characteristic set of J , then $rk f = rk h_1$ and $l = 1$, whence $\{f\}$ is also a characteristic set of the ideal J . □

5. Multivariable differential dimension polynomials and their invariants. Now we can prove the main theorem on multivariable differential dimension polynomial that generalizes the classical Kolchin's result (see [Theorem 2.1](#)).

THEOREM 5.1. *Let K be a differential field whose basic set of derivation operators Δ is a union of p disjoint finite sets ($p \geq 1$): $\Delta = \Delta_1 \cup \dots \cup \Delta_p$, where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ (m_1, \dots, m_p are some positive integers). Furthermore, let $G = K\langle \eta_1, \dots, \eta_n \rangle$ be a Δ -field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a polynomial $\Phi_\eta(t_1, \dots, t_p)$ in p variables t_1, \dots, t_p with rational coefficients such that*

- (i) $\Phi_\eta(r_1, \dots, r_p) = \text{trdeg}_K K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{Z}^p$;
- (ii) $\text{deg}_{t_i} \Phi_\eta \leq m_i$ ($i = 1, \dots, p$) and the polynomial $\Phi_\eta(t_1, \dots, t_p)$ can be written as

$$\Phi_\eta(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}, \tag{5.1}$$

where $a_{i_1 \dots i_p} \in \mathbb{Z}$ for all i_1, \dots, i_p .

PROOF. Let P be the defining Δ -ideal of the extension G/K and $\Sigma = \{A_1, \dots, A_d\}$ a characteristic set of P . Furthermore, for any $r_1, \dots, r_p \in \mathbb{N}$, let $U_{r_1 \dots r_p} = \{u \in \Theta Y \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p \text{ and either } u \text{ is not a multiple of any } u_{A_i}^{(1)}, \text{ or for every } \theta \in \Theta, A \in \Sigma \text{ such that } u = \theta u_A^{(1)}, \text{ there exists } i \in \{2, \dots, p\} \text{ such that } \text{ord}_i(\theta u_A^{(1)}) > r_i\}$. (If $p = 1$, we set $U_{r_1} = \{u \in \Theta Y \mid \text{ord}_1 u \leq r_1 \text{ and } u \text{ is not a multiple of any } u_{A_i}^{(1)}\}$.) We are going to show that the set $\bar{U}_{r_1 \dots r_p} = \{u(\eta) \mid u \in U_{r_1 \dots r_p}\}$ is a transcendence basis of the field $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p)\eta_j)$ over K .

First of all, we show that the set $\bar{U}_{r_1 \dots r_p}$ is algebraically independent over K . Let g be a polynomial in k variables ($k \in \mathbb{N}, k \geq 1$) such that $g(u_1(\eta), \dots, u_k(\eta)) = 0$ for some elements $u_1, \dots, u_k \in U_{r_1 \dots r_p}$. Then the Δ -polynomial $\bar{g} = g(u_1, \dots, u_k)$ is reduced with respect to Σ and $\bar{g} \in P$. Applying [Theorem 4.10](#) we obtain that $\bar{g} = 0$. Thus, the set $\bar{U}_{r_1 \dots r_p}$ is algebraically independent over K .

Now, we show that every element $\theta \eta_j$ ($1 \leq j \leq n, \theta \in \Theta(r_1, \dots, r_p)$) is algebraic over the field $K(\bar{U}_{r_1 \dots r_p})$. Let $\theta \eta_j \notin \bar{U}_{r_1 \dots r_p}$ (if $\theta \eta_j \in \bar{U}_{r_1 \dots r_p}$, the statement is obvious). Then $\theta y_j \notin U_{r_1 \dots r_p}$ whence θy_j is equal to some term of the form $\theta' u_{A_i}^{(1)}$ ($\theta' \in \Theta, 1 \leq i \leq d$) such that $\text{ord}_k(\theta' u_{A_i}^{(k)}) \leq r_k$ for $k = 2, \dots, p$. We represent A_i as a polynomial in $u_{A_i}^{(1)} : A_i = I_0(u_{A_i}^{(1)})^e + I_1(u_{A_i}^{(1)})^{e-1} + \dots + I_e$, where I_0, I_1, \dots, I_e do not contain $u_{A_i}^{(1)}$ (therefore, all terms in these Δ -polynomials are lower than $u_{A_i}^{(1)}$ with respect to the order $<_1$). Since $A_i \in P$,

$$A_i(\eta) = I_0(\eta) \left(u_{A_i}^{(1)}(\eta)\right)^e + I_1(\eta) \left(u_{A_i}^{(1)}(\eta)\right)^{e-1} + \dots + I_e(\eta) = 0. \tag{5.2}$$

It is clear that I_0 and $S_{A_i} = \partial A_i / \partial u_{A_i}^{(1)}$ are reduced with respect to the set Σ . By

Theorem 4.10. $I_0 \notin P$ and $S_{A_i} \notin P$ whence $I_0(\eta) \neq 0$ and $S_{A_i}(\eta) \neq 0$. If we apply the operator θ' to both sides of (5.2), the resulted equation will show that the element $\theta' u_{A_i}^{(1)}(\eta) = \theta \eta_j$ is algebraic over the field $K(\{\bar{\theta} \eta_l \mid \text{ord}_i \bar{\theta} \leq r_i \ (1 \leq i \leq p, 1 \leq l \leq n)$ and $\bar{\theta} y_l <_1 \theta' u_{A_i}^{(1)} = \theta y_j\}$). Now, the induction on the set ΘY ordered by the relation $<_1$ completes the proof of the fact that $\bar{U}_{r_1 \dots r_p}(\eta)$ is a transcendence basis of the field $K(\bigcup_{j=1}^n \Theta(r_1, \dots, r_p) \eta_j)$ over K .

Let $U_{r_1 \dots r_p}^{(1)} = \{u \in \Theta Y \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p \text{ and } u \neq \theta u_{A_j}^{(1)} \text{ for any } \theta \in \Theta; j = 1, \dots, d\}$ and $U_{r_1 \dots r_p}^{(2)} = \{u \in \Theta Y \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p \text{ and there exists at least one pair } i, j \ (1 \leq i \leq p, 1 \leq j \leq d) \text{ such that } u = \theta u_{A_j}^{(1)} \text{ and } \text{ord}_i(\theta u_{A_j}^{(1)}) > r_i\}$. (If $p = 1$, then we set $U_{r_1}^{(2)} = \emptyset$.) Clearly, $U_{r_1 \dots r_p} = U_{r_1 \dots r_p}^{(1)} \cup U_{r_1 \dots r_p}^{(2)}$ and $U_{r_1 \dots r_p}^{(1)} \cap U_{r_1 \dots r_p}^{(2)} = \emptyset$.

By Theorem 3.2, there exists a numerical polynomial $\omega(t_1, \dots, t_p)$ in p variables t_1, \dots, t_p such that $\omega(r_1, \dots, r_p) = \text{Card } U_{r_1 \dots r_p}^{(1)}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{Z}^p$ and $\text{deg}_{t_i} \omega \leq m_i \ (i = 1, \dots, p)$. Thus, in order to complete the proof of the theorem, we need to show that there exists a numerical polynomial $\phi(t_1, \dots, t_p)$ in p variables t_1, \dots, t_p such that $\phi(r_1, \dots, r_p) = \text{Card } U_{r_1 \dots r_p}^{(2)}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{Z}^p$ and $\text{deg}_{t_i} \phi \leq m_i \ (i = 1, \dots, p)$.

Let $\text{ord}_i u_{A_j}^{(1)} = a_{ij}$ and $\text{ord}_i u_{A_j}^{(i)} = b_{ij}$ for $i = 1, \dots, p; j = 1, \dots, d$ (clearly, $a_{ij} \leq b_{ij}$ and $a_{1j} = b_{1j}$ for $i = 1, \dots, p; j = 1, \dots, d$). Furthermore, for any $q = 1, \dots, p$ and for any integers k_1, \dots, k_q such that $1 \leq k_1 < \dots < k_q \leq p$, let $V_{j;k_1, \dots, k_q}(r_1, \dots, r_p) = \{\theta u_{A_j}^{(1)} \mid \text{ord}_i \theta \leq r_i - a_{ij} \text{ for } i = 1, \dots, p \text{ and } \text{ord}_k \theta > r_k - b_{kj} \text{ if and only if } k \text{ is equal to one of the numbers } k_1, \dots, k_q\}$. Then $\text{Card } V_{j;k_1, \dots, k_q}(r_1, \dots, r_p) = \phi_{j;k_1, \dots, k_q}(r_1, \dots, r_p)$, where $\phi_{j;k_1, \dots, k_q}(t_1, \dots, t_p)$ is a numerical polynomial in p variables t_1, \dots, t_p defined by the formula

$$\begin{aligned} \phi_{j;k_1, \dots, k_q}(t_1, \dots, t_p) &= \binom{t_1 + m_1 - a_{1j}}{m_1} \dots \binom{t_{k_1-1} + m_{k_1-1} - a_{k_1-1,j}}{m_{k_1-1}} \\ &\quad \times \left[\binom{t_{k_1} + m_{k_1} - a_{k_1,j}}{m_{k_1}} - \binom{t_{k_1} + m_{k_1} - b_{k_1,j}}{m_{k_1}} \right] \\ &\quad \times \binom{t_{k_1+1} + m_{k_1+1} - a_{k_1+1,j}}{m_{k_1+1}} \dots \binom{t_{k_q-1} + m_{k_q-1} - a_{k_q-1,j}}{m_{k_q-1}} \quad (5.3) \\ &\quad \times \left[\binom{t_{k_q} + m_{k_q} - a_{k_q,j}}{m_{k_q}} - \binom{t_{k_q} + m_{k_q} - b_{k_q,j}}{m_{k_q}} \right] \\ &\quad \times \dots \binom{t_p + m_p - a_{pj}}{m_p}. \end{aligned}$$

(By Theorem 3.2(iii), $\text{Card } \{\theta \in \Theta \mid \text{ord}_1 \theta \leq s_1, \dots, \text{ord}_p \theta \leq s_p\} = \prod_{i=1}^p \binom{s_i + m_i}{m_i}$ for any $s_1, \dots, s_p \in \mathbb{N}$.) Clearly, $\text{deg}_{t_i} \phi_{j;k_1, \dots, k_q} \leq m_i$ for $i = 1, \dots, p$.

Now, for any $j = 1, \dots, d$, let $V_j(r_1, \dots, r_p) = \{\theta u_{A_j}^{(1)} \mid \text{ord}_i \theta \leq r_i - a_{ij} \text{ for } i = 1, \dots, p \text{ and there exists } k \in \mathbb{N}, 1 \leq k \leq p, \text{ such that } \text{ord}_k \theta > r_k - b_{kj}\}$. Applying the combinatorial principle of inclusion and exclusion we obtain that $\text{Card } V_j(r_1, \dots, r_p) =$

$\phi_j(r_1, \dots, r_p)$, where

$$\begin{aligned} \phi_j(t_1, \dots, t_p) &= \sum_{k_1=1}^p \phi_{j;k_1}(t_1, \dots, t_p) - \sum_{1 \leq k_1 < k_2 \leq p} \phi_{j;k_1, k_2}(t_1, \dots, t_p) + \dots \\ &+ (-1)^{\nu-1} \sum_{1 \leq k_1 < \dots < k_\nu \leq p} \phi_{j;k_1, \dots, k_\nu}(t_1, \dots, t_p) + \dots \\ &+ (-1)^{p-1} \phi_{j;1, 2, \dots, p}(t_1, \dots, t_p), \end{aligned} \tag{5.4}$$

and $\text{Card } U_{r_1 \dots r_p}^{(2)}$ is an alternating sum of the terms $\text{Card } \bigcap_{v=1}^s V_{j_v}(r_1, \dots, r_p)$ ($1 \leq s \leq d$, $1 \leq j_1 < \dots < j_s \leq d$). Thus, it is sufficient to prove that $\text{Card } \bigcap_{v=1}^s V_{j_v}(r_1, \dots, r_p) = \phi_{j_1, \dots, j_s}(r_1, \dots, r_p)$, where $\phi_{j_1, \dots, j_s}(t_1, \dots, t_p)$ is a numerical polynomial such that $\text{deg}_{t_i} \phi_{j_1, \dots, j_s} \leq m_i$ for $i = 1, \dots, p$. If $\bigcap_{v=1}^s V_{j_v}(r_1, \dots, r_p) \neq \emptyset$ (so that $u_{A_{j_1}}^{(1)}, \dots, u_{A_{j_s}}^{(1)}$ contain the same Δ -indeterminate y_i), then let $v(j_1, \dots, j_s) = \text{lcm}(u_{A_{j_1}}^{(1)}, \dots, u_{A_{j_s}}^{(1)})$ and let the elements $y_1, \dots, y_s \in \Theta$ be defined by the conditions $v(j_1, \dots, j_s) = y_\nu u_{A_{j_\nu}}$ ($\nu = 1, \dots, s$). In this case, $\bigcap_{v=1}^s V_{j_v}(r_1, \dots, r_p) = \{u = \theta v(j_1, \dots, j_s) \mid \text{ord}_i u \leq r_i \text{ for } i = 1, \dots, p, \text{ and for any } l = 1, \dots, s, \text{ there exists at least one index } k \in \{2, \dots, p\} \text{ such that } \text{ord}_k(\theta y_l u_{A_{j_l}}^{(k)}) > r_k\}$. Setting $c_{j_1, \dots, j_s}^{(i)} = \text{ord}_i v(j_1, \dots, j_s)$ ($1 \leq i \leq p$) and applying the principle of inclusion and exclusion once again, we obtain that $\text{Card } \bigcap_{v=1}^s V_{j_v}(r_1, \dots, r_p)$ is an alternating sum of terms of the form $\text{Card } W(j_1, \dots, j_s; k_{11}, k_{12}, \dots, k_{1q_1}, k_{21}, \dots, k_{sq_s}; r_1, \dots, r_p)$, where $W(j_1, \dots, j_s; k_{11}, k_{12}, \dots, k_{1q_1}, k_{21}, \dots, k_{sq_s}; r_1, \dots, r_p) = \{\theta \in \Theta \mid \text{ord}_i \theta \leq r_i - c_{j_1, \dots, j_s}^{(i)}$ for $i = 1, \dots, p$, and for any $l = 1, \dots, s, \text{ord}_k \theta > r_k - c_{j_1, \dots, j_s}^{(k)} + a_{k j_l} - b_{k j_l}$ if and only if k is equal to one of the numbers $k_{l1}, \dots, k_{l q_l}\}$ (q_1, \dots, q_s are some positive integers from the set $\{1, \dots, p\}$ and $\{k_{i\nu} \mid 1 \leq i \leq s, 1 \leq \nu \leq q_s\}$ is a family of integers such that $2 \leq k_{i1} < k_{i2} < \dots < k_{i q_i} \leq p$ for $i = 1, \dots, s$).

Thus, it is sufficient to show that $\text{Card } W(j_1, \dots, j_s; k_{11}, \dots, k_{sq_s}; r_1, \dots, r_p) = \psi_{k_{11}, \dots, k_{sq_s}}^{j_1, \dots, j_s}(r_1, \dots, r_p)$ where $\psi_{k_{11}, \dots, k_{sq_s}}^{j_1, \dots, j_s}(t_1, \dots, t_p)$ is a numerical polynomial in p variables t_1, \dots, t_p such that $\text{deg}_i \psi_{k_{11}, \dots, k_{sq_s}}^{j_1, \dots, j_s} \leq m_i$ ($i = 1, \dots, p$). But this is almost evident: as in the process of evaluation of $\text{Card } V_{j_1, \dots, j_s}(r_1, \dots, r_p)$ (when we used [Theorem 3.2\(iii\)](#) to obtain formula (5.3)), we see that $\text{Card } W(j_1, \dots, j_s; k_{11}, \dots, k_{sq_s}; r_1, \dots, r_p)$ is a product of terms of the form $\binom{r_k + m_k - c_{j_1, \dots, j_s}^{(k)}}{m_k}^{-S_k}$ (such a term corresponds to an integer k such that $1 \leq k \leq p$ and $k \neq k_{i\nu}$ for any $i = 1, \dots, s, \nu = 1, \dots, q_s$; the number S_k is defined as $\max\{b_{k j_l} - a_{k j_l} \mid 1 \leq l \leq s\}$) and terms of the form $\left[\binom{r_k + m_k - c_{j_1, \dots, j_s}^{(k)}}{m_k} - \binom{r_k + m_k - c_{j_1, \dots, j_s}^{(k)} - T_k}{m_k} \right]$ (such a term appears in the product if k is equal to some $k_{i\nu}$ ($1 \leq i \leq s, 1 \leq \nu \leq q_s$)). In this case, if $k_{i_1 \nu_1}, \dots, k_{i_e \nu_e}$ are all elements of the set $\{k_{i\nu} \mid 1 \leq i \leq s, 1 \leq \nu \leq q_s\}$ that are equal to k ($1 \leq e \leq s, 1 \leq i_1 < \dots < i_e \leq s$), then T_k is defined as $\min\{b_{k j_{i_\lambda}} - a_{k j_{i_\lambda}} \mid 1 \leq \lambda \leq e\}$.

The appropriate numerical polynomial $\psi_{k_{11}, \dots, k_{sq_s}}^{j_1, \dots, j_s}(t_1, \dots, t_p)$ is a product of p “elementary” numerical polynomials f_1, \dots, f_p where f_k ($1 \leq k \leq p$) is either a polynomial of the form $\binom{t_k + m_k - c_{j_1, \dots, j_s}^{(k)}}{m_k}^{-S_k}$ or apolynomial of the form $\left[\binom{t_k + m_k - c_{j_1, \dots, j_s}^{(k)}}{m_k} - \binom{t_k + m_k - c_{j_1, \dots, j_s}^{(k)} - T_k}{m_k} \right]$. Since the degree of such a product with respect to any variable t_i ($1 \leq i \leq p$) does not exceed m_i , this completes the proof of the theorem. \square

DEFINITION 5.2. Numerical polynomial $\Phi_\eta(t_1, \dots, t_p)$, whose existence is established by [Theorem 5.1](#), is called a dimension polynomial of the differential field extension $G = K\langle \eta_1, \dots, \eta_n \rangle$ associated with the given system of Δ -generators $\eta = \{\eta_1, \dots, \eta_n\}$ and with the given partition of the basic set of derivation operators Δ into p disjoint subsets $\Delta_1, \dots, \Delta_p$.

In what follows, we will consider different lexicographic orders on a set \mathbb{N}^p (p is a positive integer). For any permutation (j_1, \dots, j_p) of the set $\{1, \dots, p\}$, we define the corresponding lexicographic order \leq_{j_1, \dots, j_p} on \mathbb{N}^p as follows: $(r_1, \dots, r_p) \leq_{j_1, \dots, j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbb{N}$, $1 \leq k \leq p - 1$ such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$. If Σ is a finite subset of \mathbb{N}^p , then an element $e \in \Sigma$ will be called a *lex-maximal* element of this set, if e is the maximal element of Σ with respect to one of the $p!$ lexicographic orders \leq_{j_1, \dots, j_p} (where (j_1, \dots, j_p) is a permutation of the set $\{1, \dots, p\}$). The set of all lex-maximal elements of a finite set $\Sigma \subseteq \mathbb{N}^p$ will be denoted by Σ' . For example, if $\Sigma = \{(3, 0, 2), (2, 1, 1), (0, 1, 4), (1, 0, 3), (1, 1, 6), (3, 1, 0), (1, 2, 0)\} \subseteq \mathbb{N}^3$, then $\Sigma' = \{(3, 0, 2), (3, 1, 0), (1, 1, 6), (1, 2, 0)\}$.

THEOREM 5.3. *Let K be a differential field whose basic set Δ is a union of p disjoint finite sets: $\Delta = \Delta_1 \cup \dots \cup \Delta_p$ ($p \geq 1$). Let G be a Δ -field extension of K with the finite set of Δ -generators $\eta = \{\eta_1, \dots, \eta_n\}$ and let*

$$\Phi_\eta(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p} \tag{5.5}$$

be the corresponding dimension polynomial ($m_i = \text{Card} \Delta_i$ for $i = 1, \dots, p$). Furthermore, let

$$E_\eta = \{(i_1 \cdots i_p) \in \mathbb{N}^p(m_1, \dots, m_p) \mid a_{i_1 \dots i_p} \neq 0\}. \tag{5.6}$$

Then the elements $(k_1, \dots, k_p) \in E'_\eta$ and the corresponding coefficients $a_{k_1 \dots k_p}$ do not depend on the choice of the system of Δ -generators η .

PROOF. Let $\zeta = \{\zeta_1, \dots, \zeta_q\}$ be another system of Δ -generators of G over K and let

$$\Phi_\zeta(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} b_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p} \tag{5.7}$$

be the dimension polynomial of our Δ -field extension associated with the system of generators ζ . Then there exist positive integers s_1, \dots, s_p such that $\eta_i \in K(\bigcup_{j=1}^q \Theta(s_1, \dots, s_p) \zeta_j)$ and $\zeta_k \in K(\bigcup_{j=1}^n \Theta(s_1, \dots, s_p) \eta_j)$ ($1 \leq i \leq n$, $1 \leq k \leq q$), whence $\Phi_\eta(r_1, \dots, r_p) \leq \Phi_\zeta(r_1 + s_1, \dots, r_p + s_p)$ and $\Phi_\zeta(r_1, \dots, r_p) \leq \Phi_\eta(r_1 + s_1, \dots, r_p + s_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{Z}^p$. It follows that for any element $(k_1, \dots, k_p) \in E'_\eta$, the term $\binom{t_1 + k_1}{k_1} \cdots \binom{t_p + k_p}{k_p}$ appears in $\Phi_\eta(t_1, \dots, t_p)$ and $\Phi_\zeta(t_1, \dots, t_p)$ with the same coefficient $a_{k_1 \dots k_p}$. This completes the proof of the theorem. \square

The following example shows that a dimension polynomial of a finitely generated differential field extension associated with some partition of the basic set of derivation operators can carry more differential birational invariants of the extension than the classical Kolchin differential dimension polynomial.

EXAMPLE 5.4. Let K be a differential field with a basic set of derivation operators $\Delta = \{\delta_1, \delta_2\}$ and let L be a Δ -field extension of K generated by a single Δ -generator η with the defining equation

$$\delta_1^a \delta_2^b \eta + \delta_2^{a+b} \eta = 0, \tag{5.8}$$

where a and b are some positive integers. In other words, $L = K\langle \eta \rangle$ is Δ -isomorphic to the quotient field of the factor ring $K\{\mathcal{Y}\}/P$, where P is the Δ -ideal of the ring of Δ -polynomials $K\{\mathcal{Y}\}$ generated by the Δ -polynomial $f(\mathcal{Y}) = \delta_1^a \delta_2^b \mathcal{Y} + \delta_2^{a+b} \mathcal{Y}$. Let $\omega_{\eta/K}(t)$ be the Kolchin differential dimension polynomial of the extension L/K and let $\Phi_\eta(t_1, t_2)$ be the dimension polynomial of this extension associated with the partition $\Delta = \{\delta_1\} \cup \{\delta_2\}$. By [Theorem 4.11](#), $\{f(\mathcal{Y})\}$ is a characteristic set of the ideal P . Using the notation of the proof of [Theorem 5.1](#), we obtain that $U_{r_1 r_2}^{(1)} = \{u \in \Theta Y \mid \text{ord}_1 u \leq r_1, \text{ord}_2 u \leq r_2 \text{ and } \delta_1^a \delta_2^b \mathcal{Y} \nmid u\}$ (so that $\text{Card } U_{r_1 r_2}^{(1)} = (r_1 + 1)(r_2 + 1) - (r_1 + 1 - a)(r_2 + 1 - b) = br_1 + ar_2 + a + b - ab$) and $U_{r_1 r_2}^{(2)} = \{u = \theta \delta_1^a \delta_2^b \mathcal{Y} \mid \theta \in \Theta(r_1 - a, r_2 - b) \setminus \Theta(r_1 - a, r_2 - (a + b))\}$ (so that $\text{Card } U_{r_1 r_2}^{(2)} = a(r_1 - a + 1)$). Since $\Phi_\eta(r_1, r_2) = \text{Card } U_{r_1 r_2}^{(1)} + \text{Card } U_{r_1 r_2}^{(2)}$ for all sufficiently large $(r_1, r_2) \in \mathbb{Z}^2$, we obtain that

$$\Phi_\eta(t_1, t_2) = (a + b)t_1 + at_2 + 2a + b - ab - a^2. \tag{5.9}$$

The computation of the Kolchin differential dimension polynomial with the help of the formula in [\[7, Chapter II, Theorem 6\(d\)\]](#) leads to the following result:

$$\omega_{\eta/K}(t) = \binom{t+2}{2} - \binom{t+2-(a+b)}{2} = (a+b)t - \frac{(a+b)(a+b-3)}{2}. \tag{5.10}$$

Comparing the polynomials $\omega_{\eta/K}(t)$ and $\Phi_\eta(t_1, t_2)$, we see that the first polynomial carries two differential birational invariants, its degree 1 and the leading coefficient $a + b$, while $\Phi_\eta(t_1, t_2)$ carries three such invariants, its total degree 1, $a + b$, and a (therefore, a and b are uniquely determined by the polynomial Φ_η). Thus, $\Phi_\eta(t_1, t_2)$ gives both parameters a and b of [\(5.8\)](#) while $\omega_{\eta/K}(t)$ gives just the sum of the parameters.

REFERENCES

[1] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Ph.D. thesis, University of Innsbruck, Austria, 1965.

[2] A. Einstein, *Supplement to Appendix II of "The Meaning of Relativity"*, 4th ed., Princeton University Press, New Jersey, 1953. [MR 15,357e](#).

[3] C. G. J. Jacobi, *Gesammelte Werke*, vol. 5, G. Reimer, Berlin, 1890 (German).

[4] J. Johnson, *Differential dimension polynomials and a fundamental theorem on differential modules*, *Amer. J. Math.* **91** (1969), 239–248. [MR 39#186](#). [Zbl 179.34303](#).

[5] ———, *Kähler differentials and differential algebra in arbitrary characteristic*, *Trans. Amer. Math. Soc.* **192** (1974), 201–208. [MR 49#263](#). [Zbl 288.12104](#).

[6] E. R. Kolchin, *The notion of dimension in the theory of algebraic differential equations*, *Bull. Amer. Math. Soc.* **70** (1964), 570–573. [MR 29#5816](#). [Zbl 144.03702](#).

[7] ———, *Differential Algebra and Algebraic Groups*, *Pure and Applied Mathematics*, vol. 54, Academic Press, New York, 1973. [MR 58#27929](#). [Zbl 264.12102](#).

[8] M. V. Kondrat'eva, A. B. Levin, A. V. Mikhal'ev, and E. V. Pankrat'ev, *Computation of dimension polynomials*, *Internat. J. Algebra Comput.* **2** (1992), no. 2, 117–137. [MR 93k#12012](#). [Zbl 774.12007](#).

- [9] ———, *Differential and Difference Dimension Polynomials*, Mathematics and its Applications, vol. 461, Kluwer Academic Publishers, Dordrecht, 1999. [MR 2001c:12006](#). [Zbl 930.12005](#).
- [10] A. B. Levin, *Characteristic polynomials of inverse differential modules and some properties of inverse differential dimension*, Russian Math. Surveys **35** (1980), no. 1, 217–218. [Zbl 459.12017](#).
- [11] A. B. Levin and A. V. Mikhalëv, *A differential dimension polynomial and the rigidity of a system of differential equations*, Computable Invariants in the Theory of Algebraic Systems, Akad. Nauk SSSR Sibirsk. Otdel. Vychisl. Tsent, Novosibirsk, 1987, pp. 58–66 (Russian). [MR 90b#12001](#). [Zbl 722.12004](#).
- [12] A. V. Mikhalëv and E. V. Pankrat'ev, *Differential dimension polynomial of a system of differential equations*, Algebra, Collection of Papers, Moscow State Univ. Press, Moscow, 1980, pp. 57–67 (Russian). [Zbl 0498.12027](#).
- [13] W. Sit, *Well-ordering of certain numerical polynomials*, Trans. Amer. Math. Soc. **212** (1975), 37–45. [MR 53#10776](#). [Zbl 295.12103](#).
- [14] ———, *Differential dimension polynomials of finitely generated extensions*, Proc. Amer. Math. Soc. **68** (1978), no. 3, 251–257. [MR 81f#12017](#). [Zbl 391.12012](#).
- [15] W. Sit and J. Johnson, *On the differential transcendence polynomials of finitely generated differential field extensions*, Amer. J. Math. **101** (1979), no. 6, 1249–1263. [MR 80m:12029](#). [Zbl 434.12011](#).

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