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RESULTS ON COMMON FIXED POINTS

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ABSTRACT. We establish common fixed point theorems related with families of self-mappings on metric spaces. Our results extend, improve, and unify the results due to Fisher (1977, 1978, 1979, 1981, 1984), Jungck (1988), and Ohta and Nikaido (1994).

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1. Introduction. Let w and $\mathbb{N}$ denote the sets of nonnegative integers and positive integers, respectively. For $t \in [0,\infty)$, $[t]$ denotes the largest integer not exceeding t. Let f and g be mappings from a metric space (X,d) into itself and $C_f = \{h : h : X \to X \text{ and } hf = fh\}$. For $x,y \in X$ and $A \subseteq X$, define $O_f(x) = \{f^n x : n \in w\}$, $O_f(x,y) = O_f(x) \cup O_f(y)$, $O_{f,g}(x) = \{f^i g^j x : i,j \in w\}$, $O_{f,g}(x,y) = O_{f,g}(x) \cup O_{f,g}(y)$, and $\delta(A) = \sup\{d(x,y) : x,y \in A\}$. Let i_X denote the identity mapping on X. It is easy to see that $\{f^n : n \in w\} \subseteq C_f$. Let $\Phi = \{\varphi : \varphi : [0,\infty) \to [0,\infty)$ is upper semicontinuous and nondecreasing and $\varphi(t) < t$ for $t > 0\}$. The following definition and lemmas were introduced by Fisher [8], Singh and Meade [13], and Jungck [10], respectively.

Definition 1.1 (see [8]). Let $A \subseteq X$ and $A_n \subseteq X$ for all $n \in \mathbb{N}$. The sequence $\{A_n\}_{n\in\mathbb{N}}$ is said to converge to A if

(i) each point $a \in A$ is the limit of some convergent sequence $\{a_n\}_{n\in\mathbb{N}}$, where $a_n \in A_n$ for all $n \in \mathbb{N}$;

(ii) for arbitrary $\varepsilon > 0$, there exists an integer k such that $A_n \subseteq A_\varepsilon$ for $n > k$, where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 1.2 (see [13]). *Let $\varphi \in \Phi$, then* (a) $\lim_{n\to\infty} \varphi^n(t) = 0$ *for all* $t > 0$ *and* (b) $t = 0$ *provided that* $t \le \varphi(t)$ *for some* $t \ge 0$.

Lemma 1.3 (see [10]). *Let f and g be commuting self-mappings of a compact metric space (X,d) such that gf is continuous. If $A = \bigcap_{n\in\mathbb{N}} (gf)^n X$, then*

(i) *$hA \subseteq A$ for all $h \in C_{gf}$;*

(ii) *$A = fA = gA \neq \phi$;*

(iii) *A is compact.*

In recent years, a number of generalizations of a well-known contraction mapping principles due to Banach have appeared in the literature (cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

First we list the following general conditions.

(1) There exists $\varphi \in \Phi$ and $p,q,m,n \in w$ with $p+q, m+n \in \mathbb{N}$ such that

$$d(f^p g^q x, f^m g^n y) \le \varphi\left(\delta\left(\bigcup_{h \in C_g \cup C_f} hO_{g,f}(x,y)\right)\right) \tag{1.1}$$

for all $x,y \in X$.

(2) There exist $p,q,m,n \in w$ with $p+q, m+n \in \mathbb{N}$ such that

$$d(f^p g^q x, f^m g^n y) < \delta\left(\bigcup_{h \in C_{gf}} hO_{gf}(x,y)\right) \tag{1.2}$$

for all $x,y \in X$ with $f^p g^q x \ne f^m g^n y$.

In the literature of fixed point theory there exist conditions which are special cases of (1) or (2). Now we list below some of the contractive mappings for which various fixed point theorems have been established.

(3) (See [6].) There exists $r \in [0,1)$ and $p,q,m,n \in w$ with $p+q, m+n \in \mathbb{N}$ such that

$$\begin{aligned} &d(f^p g^q x, f^m g^n y) \\ &\quad \le r \cdot \max\{d(f^r g^{r'} x, f^s g^{s'} y), \\ &\qquad d(f^r g^{r'} x, f^t g^{t'} x), d(f^s g^{s'} x, f^i g^{i'} y): \\ &\qquad 0 \le r,\ t \le p,\ 0 \le r',\ t' \le q,\ 0 \le s,\ i \le m,\ 0 \le s',\ i' \le n\} \end{aligned} \tag{1.3}$$

for all $x,y \in X$.

(4) (See [6].) There exist $p,q,m,n \in w$ with $p+q, m+n \in \mathbb{N}$ such that

$$\begin{aligned} &d(f^p g^q x, f^m g^n y) \\ &\quad < \max\{d(f^r g^{r'} x, f^s g^{s'} y), \\ &\qquad d(f^r g^{r'} x, f^t g^{t'} x), d(f^s g^{s'} x, f^i g^{i'} y): \\ &\qquad 0 \le r,\ t \le p,\ 0 \le r',\ t' \le q,\ 0 \le s,\ i \le m,\ 0 \le s',\ i' \le n\} \end{aligned} \tag{1.4}$$

for all $x,y \in X$ for which the right-hand side of the inequality is positive.

(5) (See [11].) There exist $k \in \mathbb{N}$ and $r \in [0,1)$ such that

$$d(f^k x, f^k y) \le r\delta(O_f(x,y)) \tag{1.5}$$

for all $x,y \in X$.

(6) (See [10].) For all $x,y \in X$, $d(fx,gy) < \delta(ht : t \in \{x,y\},\ h \in C_{gf})$ with $fx \ne gy$.

In this note, we prove common fixed point theorems for $C_f \cap C_g$ and C_{gf} in bounded complete metric spaces and compact metric spaces. Our results extend, improve and unify the corresponding results in [1, 2, 3, 4, 5, 6, 7, 9, 10, 11].

2. Common fixed point theorems.

Our main results are as follows.

Theorem 2.1. *Let f and g be commuting mappings from a bounded complete metric space (X,d) into itself. If f and g are continuous and satisfy (1), then*

(i) *f and g have a unique common fixed point $u \in X$ which is also a unique common fixed point of $C_f \cap C_g$;*

(ii) $d(f^{i+a}g^{i+b}x,u) \le \varphi^{[i/k]}(\delta(X))$ *for all* $a,b \in \{0,1\}$ *and* $i \in \mathbb{N}$*, where* $k = \max\{p,q,m,n\}$*;*

(iii) $\lim_{i\to\infty} f^{i+a}g^{i+b}x = u$ *for all* $a,b \in \{0,1\}$*;*

(iv) $\{f^i g^i X\}_{i\in\mathbb{N}}$ *converges to* $\{u\}$*.*

Proof. For any $i \in w$ and $x,y \in X$, it follows from (1) that

$$\begin{aligned}
&d(f^{i+k}g^{i+k}x, f^{i+k}g^{i+k}y) \\
&\quad\le \varphi\bigg(\delta\bigg(\bigcup_{h\in C_f\cap C_g} hO_{f,g}(f^{i+k-p}g^{i+k-q}x, f^{i+k-m}g^{i+k-n}y)\bigg)\bigg) \\
&\quad\le \varphi\bigg(\delta\bigg(\bigcup_{h\in C_f\cap C_g} hO_{f,g}(f^i g^i x, f^i g^i y)\bigg)\bigg) \\
&\quad\le \varphi\bigg(\delta\bigg(\bigcup_{h\in C_f\cap C_g} hf^i g^i X\bigg)\bigg) \\
&\quad= \varphi\bigg(\delta\bigg(\bigcup_{h\in C_f\cap C_g} f^i g^i hX\bigg)\bigg) \\
&\quad= \varphi(\delta(f^i g^i X))
\end{aligned} \tag{2.1}$$

which implies that

$$\delta(f^{i+k}g^{i+k}X) = \sup_{x,y\in X} d(f^{i+k}g^{i+k}x, f^{i+k}g^{i+k}y) \le \varphi(\delta(f^i g^i X)) \tag{2.2}$$

for all $i \in w$. We can write $i = jk+t$ uniquely for some $j,k \in w$ with $t < k$. Thus

$$\delta(f^i g^i X) \le \varphi(\delta(f^{(j-1)k+t}g^{(j-1)k+t}X)) \le \varphi^j(\delta(f^t g^t X)) \le \varphi^j(\delta(X)). \tag{2.3}$$

It follows from the boundedness of X, Lemma 1.2 and (2.3) that

$$\lim_{i\to\infty} \delta(f^i g^i X) = \lim_{j\to\infty} \varphi^j(\delta(X)). \tag{2.4}$$

Since $d(f^i g^i x, f^{i+t}g^{i+t}x) \le \delta(f^i g^i X)$ for all $i,t \in w$ and $x \in X$, $\{f^i g^i x\}_{i\in\mathbb{N}}$ is a Cauchy sequence and converges to some $u \in X$ by completeness of X. For any $i,c \in w$, $a,b \in \{0,1\}$ and $x \in X$, by (2.3) and (2.4) we have

$$d(f^{i+a}g^{i+b}x, f^{i+c}g^{i+c}x) \le \delta(f^i g^i X) \le \varphi^{[i/k]}(\delta(X)). \tag{2.5}$$

Letting c tend to infinity, we get

$$d(f^{i+a}g^{i+b}x, u) \le \varphi^{[i/k]}(\delta(X)) \tag{2.6}$$

for $x \in X$. The continuity of f and g, and (2.4) and (2.6) ensure that

$$u = \lim_{i\to\infty} f^i g^i x = \lim_{i\to\infty} f^a g^b f^i g^i x = f^a g^b u \tag{2.7}$$

for all $a,b \in \{0,1\}$ and $x \in X$. That is, $fu = gu = u$.

Suppose that f and g have another common fixed point $v \in X$. It follows from (2.4) that

$$d(u,v) \le \delta(f^i g^i X) \le \varphi^{[i/k]}(\delta(X)) \longrightarrow 0 \quad \text{as } i \longrightarrow \infty. \tag{2.8}$$

That is, $u = v$. Therefore f and g have a unique common fixed point. Note that $fhu = hfu = hu = hgu = ghu$ for all $h \in C_f \cap C_g$. By the uniqueness of common fixed point of f and g, we have $hu = u$ for all $h \in C_f \cap C_g$. Since $f, g \in C_f \cap C_g$, it follows that u is a unique common fixed point of $C_f \cap C_g$.

Let $\varepsilon > 0$. In view of (2.4), there exists $c \in \mathbb{N}$ such that

$$\delta(f^i g^i X) \le \varphi^{[i/k]}(\delta(X)) < \frac{1}{2}\varepsilon. \tag{2.9}$$

for all $i > c$. Note that $u \in f^i g^i X$ for all $i \in w$. Thus, for any $i > c$ we have

$$f^i g^i X \subseteq B(u,\varepsilon) = \{x \in X : d(u,x) < \varepsilon\}. \tag{2.10}$$

Therefore $\{f^i g^i X\}_{i\in\mathbb{N}}$ converges to $\{u\}$. This completes the proof. □

Remark 2.2. [6, Theorem 2] is a special case of Theorem 2.1.

As an immediate consequence of Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let f and g be commuting mappings from a bounded complete metric space (X,d) into itself. If f and g are continuous and satisfy the inequality*

$$d(f^p g^q x, f^m g^n y) \le r\delta\left(\bigcup_{h\in C_f\cap C_g} hO_{f,g}(x,y)\right) \tag{2.11}$$

for all $x, y \in X$, where $p, q, m, n \in w$, $p+q, m+n \in \mathbb{N}$ and $r \in [0,1)$. Then (i), (iii), and (iv) of Theorem 2.1 and the following (v) hold:

(v) *$d(f^{i+a} g^{i+b} x, u) \le r^{[i/k]}\delta(X)$ for all $a, b \in \{0,1\}$ and $i \in \mathbb{N}$, where $k = \max\{p,q,m,n\}$.*

Remark 2.4. In case $p = m$, $g = i_X$, Corollary 2.3 reduces to a result which generalizes [11, Theorem 3].

Theorem 2.5. *Let f and g be commuting mappings from a compact metric space (X,d) into itself such that gf is continuous. If (2) is satisfied, then f and g have a unique common fixed point $u \in X$. Moreover, $u = hu$ for all $h \in C_{gf}$.*

Proof. Let $A = \bigcap_{n\in\mathbb{N}} (gf)^n X$. It follows from Lemma 1.3 that $A = fA = gA \ne \emptyset$ and that A is compact. We claim that $A = \{u\}$ for some $u \in X$. Otherwise $\delta(A) > 0$. By the compactness of A there exists distinct $u, v \in A$ such that $\delta(A) = d(u,v)$. Clearly, we can find $x, y \in A$ such that $f^p g^q x = u$ and $f^m g^n y = v$. Using (2) and Lemma 1.3 we have

$$\delta(A) = d(f^p g^q x, f^m g^n y) < \delta\left(\bigcup_{h\in C_{gf}} hO_{gf}(x,y)\right) \le \delta\left(\bigcup_{h\in C_{gf}} hA\right) \le \delta(A) \tag{2.12}$$

which is a contradiction. Thus $A = \{u\}$ for some $u \in X$. Lemma 1.3 ensures that $u = hu$ for all $h \in C_{gf}$. In particular, $u = fu = gu$. If f and g have another common fixed point $c \in X$. Then $c = (gf)^n c$ for all $n \in \mathbb{N}$. That is, $c \in A = \{u\}$. Hence u is the only common fixed point of f and g. This completes the proof. □

Remark 2.6. Theorem 2.5 includes [6, Theorem 5] as a special case.

As an immediate consequence of Theorem 2.5, we have the following corollary.

Corollary 2.7. *Let f and g be commuting mappings from a compact metric space (X,d) into itself. If gf is continuous and there exists $p,m \in \mathbb{N}$ such that*

$$d(f^p x, g^m y) < \delta\left(\bigcup_{h\in C_{gf}} hO_{gf}(x,y)\right) \tag{2.13}$$

for all $x,y \in X$ with $f^p x \neq g^m y$. Then the conclusion of Theorem 2.5 holds.

Corollary 2.8. *Let f be a continuous mapping from a compact metric space (X,d) into itself. Assume that there exists $p,m \in \mathbb{N}$ such that*

$$d(f^p x, f^m y) < \delta\left(\bigcup_{h\in C_{f}} hO_{f}(x,y)\right) \tag{2.14}$$

for all $x,y \in X$ with $f^p x \neq f^m y$. Then f has a unique fixed point $u \in X$ and $hu = u$ for all $h \in C_f$.

Remark 2.9. Corollary 2.7 extends, improve and unifies [10, Theorem 4.2], [9, Theorem 2], and [7, Theorem 5].

Remark 2.10. [1, Theorem 4], [2, Theorem 4], [4, Theorem 9], [3, Theorem 2], and [5, Theorem 4] are special cases of Corollary 2.8.

Remark 2.11. The following examples reveal that the condition that f and g are continuous is necessary in Theorems 2.1, 2.5, and Corollaries 2.7, 2.8.

Example 2.12. Let $X = [0,1]$ with the usual metric d. Define $f,g : X \to X$ by $g = i_X$, $f0 = 1/4$ and $fx = (1/3)x$ for all $X \in (0,1]$. Take $p = m = 2$, $q = n = 1$, $r = 1/2$, $\varphi(t) = (1/2)t$ for all $t \geq 0$. Then (X,d) is a bounded metric space, g is continuous, and f is discontinuous.

For $x,y \in (0,1]$, we have

$$\begin{aligned} d(f^p g^q x, f^m g^n y) &= \frac{1}{9}|x-y| \\ &\leq \frac{1}{2}|x-y| \leq \frac{1}{2}\delta(O_{f,g}(x,y)) \\ &\leq \varphi\left(\delta\left(\bigcup_{h\in C_f\cap C_g} hO_{f,g}(x,y)\right)\right). \end{aligned} \tag{2.15}$$

For $x = 0$, $y \in (0,1]$ or $x \in (0,1]$, $y = 0$, we have

$$\begin{aligned} d(f^p g^q x, f^m g^n y) &\leq \max\left\{\left|\frac{1}{12} - \frac{1}{9}y\right|, \left|\frac{1}{12} - \frac{1}{9}x\right|\right\} < \frac{1}{8} \\ &= \frac{1}{2}\delta(O_{f,g}(0)) \leq \varphi\left(\delta\left(\bigcup_{h\in C_f\cap C_g} hO_{f,g}(x,y)\right)\right). \end{aligned} \tag{2.16}$$

For $x = y = 0$, we have

$$d(f^p g^q x, f^m g^n y) = 0 < \frac{1}{8} = \frac{1}{2}\delta(O_{f,g}(0,0)) \le \varphi\left(\delta\left(\bigcup_{h \in C_f \cap C_g} hO_{f,g}(x,y)\right)\right). \quad (2.17)$$

That is, f and g satisfy (1) and (2.11). But f and g have no common fixed point in X.

Example 2.13. Let $(X,d), f, g, p, q, m$, and n be as in Example 2.12. It is easy to check that the conditions of Theorem 2.5 and Corollary 2.8 are satisfied except for the continuity assumption. However f has no fixed point in X.

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