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CONVERGENCE THEOREMS OF THE SEQUENCE OF ITERATES FOR A FINITE FAMILY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E. In this paper, we introduce an iteration scheme with errors in the sense of Xu (1998) generated by $\{T_j : C \to C\}_{j=1}^{r}$ as follows: $U_{n(j)} = a_{n(j)}I + b_{n(j)}T_j^n U_{n(j-1)} + c_{n(j)}u_{n(j)}$, $j = 1,2,\ldots,r$, $x_1 \in C$, $x_{n+1} = a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)}$, $n \geq 1$, where $U_{n(0)} := I$, I the identity map; and $\{u_{n(j)}\}$ are bounded sequences in C; and $\{a_{n(j)}\}$, $\{b_{n(j)}\}$, and $\{c_{n(j)}\}$ are suitable sequences in $[0,1]$. We first consider the behaviour of iteration scheme above for a finite family of asymptotically nonexpansive mappings. Then we generalize theorems of Schu and Rhoades.

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1. Introduction. Let C be a nonempty convex subset of a Banach space E. A mapping $T : C \to C$ is called *asymptotically nonexpansive with sequence* $\{k_n\}_{n=1}^{\infty}$ if $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and all $n \in \mathbb{N}$. T is called *uniformly* L*-Lipschitzian if*

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.2}$$

for all $x, y \in C$ and all $n \in \mathbb{N}$. It is clear that every asymptotically nonexpansive mapping is also uniformly L-Lipschitzian for some $L > 0$. In [7], Schu introduced *the modified Ishikawa iteration method* as

$$x_{n+1} = \alpha_n T^n(\beta_n T^n x_n + (1-\beta_n)x_n) + (1-\alpha_n)x_n, \quad n = 1,2,\ldots, \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0,1]$ and *the modified Mann iteration method* as

$$x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n = 1,2,\ldots, \tag{1.4}$$

where $\{\alpha_n\}$ is a suitable sequence in $[0,1]$.

Using the iteration method (1.4), Schu [9, Lemma 1.5] and Rhoades [6, Theorem 1] obtained the following result: *let C be a bounded closed convex subset of a uniformly* *convex Banach space E, $T : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, and $\{\alpha_n\}$ a sequence in $[0,1]$ satisfying the condition $\varepsilon \le \alpha_n \le 1-\varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Suppose that $x_1 \in C$ and that $\{x_n\}$ is given by (1.4). Then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Note that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ if and only if $\sum_{n=1}^{\infty}(k_n^s - 1) < \infty$ for some $s > 1$ (see [5, Remark 3]).

Let C be a nonempty convex subset of a Banach space E. Let $T_j : C \to C$ be a given mapping for each $j = 1,2,\dots,r$. We now introduce an iteration scheme with errors in the sense of Xu [11] generated by $T_1, T_2, \dots, T_r$ as follows: let $U_{n(0)} = I$, where I is the identity map,

$$\begin{aligned}
& U_{n(1)} = a_{n(1)}I + b_{n(1)}T_1^n U_{n(0)} + c_{n(1)}u_{n(1)},\\
& U_{n(2)} = a_{n(2)}I + b_{n(2)}T_2^n U_{n(1)} + c_{n(2)}u_{n(2)},\\
& \quad\vdots\\
& U_{n(r)} = a_{n(r)}I + b_{n(r)}T_r^n U_{n(r-1)} + c_{n(r)}u_{n(r)},\\
x_1 \in C, \quad & x_{n+1} = a_{n(r)}x_n + b_{n(r)}T_r^n U_{n(r-1)}x_n + c_{n(r)}u_{n(r)}, \quad n \ge 1.
\end{aligned} \tag{1.5}$$

Here, $\{u_{n(j)}\}_{n=1}^{\infty}$ is a bounded sequence in C for each $j = 1,2,\dots,r$, and $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$, and $\{c_{n(j)}\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying the conditions

$$a_{n(j)} + b_{n(j)} + c_{n(j)} = 1 \tag{1.6}$$

for all $n \in \mathbb{N}$ and each $j = 1,2,\dots,r$. This scheme contains the modified Mann and Ishikawa iteration methods with errors in the sense of Xu [11] (cf. [5]): for $r = 1$, our scheme reduces to Mann-Xu type iteration and for $r = 2$, $T_1 = T_2$ to Ishikawa-Xu type iteration.

In 1972, Goebel and Kirk [1] proved that if C is a bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive selfmapping T of C has a fixed point. After the existence theorem of Goebel and Kirk [1], several authors including Schu [7, 9], Rhoades [6], Huang [3] and Osilike and Aniagbosor [5] have studied methods for the iterative approximation of fixed points of asymptotically nonexpansive mappings. In this paper, we first extend the result above of [9, Lemma 1.5] and [6, Theorem 1] to the iteration scheme (1.5) and without the restrictions that C is bounded. Then, using this result, we generalize [9, Theorems 2.1, 2.2, and 2.4] and [6, Theorems 2 and 3].

In the sequel, we will need the following results.

Lemma 1.1 (see [5, Lemma 1]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad n \ge 1. \tag{1.7}$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2 (see [8, Lemma 2]). *Let $\{\beta_n\}_{n=1}^{\infty}$ and $\{\omega_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers such that for some real numbers $N_0 \geq 1$,*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \omega_n \tag{1.8}$$

for all $n \geq N_0$, where $\delta_n \in [0,1]$. If $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\sum_{n=1}^{\infty}\omega_n < \infty$, then $\lim_{n\to\infty}\beta_n = 0$.

Theorem 1.3 (see [10, Theorem 2]). *Let E be a uniformly convex Banach space and $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$ and*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{1.9}$$

for all $x, y \in B_r := \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0,1]$.

A Banach space E is said to satisfy Opial's condition [4] *if $x_n \to x$ weakly and $x \neq y$ imply*

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||. \tag{1.10}$$

Lemma 1.4 (see [2, Lemma 4]). *Let E be a uniformly convex Banach space satisfying Opial's condition and C a nonempty closed convex subset of E. Let $T : C \to C$ be an asymptotically nonexpansive mapping. Then $(I-T)$ is demiclosed at zero, that is, for each sequence $\{x_n\}$ in C, the conditions $x_n \to x$ weakly and $(I-T)x_n \to 0$ strongly imply $(I-T)x = 0$.*

2. Main results.

For abbreviation, we denote the set of fixed points of a mapping T by $F(T)$, and now prove the following results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T_j : C \to C$ an asymptotically nonexpansive mapping with sequence $\{k_{n(j)}\}_{n=1}^{\infty}$ for each $j = 1,2,\ldots,r$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, where $k_n := \max_{1\leq j\leq r}\{k_{n(j)}\} \geq 1$ and $\cap_{j=1}^{r}F(T_j) \neq \emptyset$. Let $\{u_{n(j)}\}_{n=1}^{\infty}$ be a bounded sequence in C for each $j = 1,2,\ldots,r$ and let $\{a_{n(j)}\}_{n=1}^{\infty}$, $\{b_{n(j)}\}_{n=1}^{\infty}$, and $\{c_{n(j)}\}_{n=1}^{\infty}$ be sequences in $[0,1]$ satisfying the conditions:*

(i) *$a_{n(j)} + b_{n(j)} + c_{n(j)} = 1$ for all $n \in \mathbb{N}$ and each $j = 1,2,\ldots,r$;*

(ii) *$\sum_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1,2,\ldots,r$;*

(iii) *$0 < a \leq \alpha_{n(j)} \leq b < 1$ for all $n \in \mathbb{N}$, each $j = 1,2,\ldots,r$, and some constants a, b, where $\alpha_{n(j)} := b_{n(j)} + c_{n(j)}$.*

Suppose that $\{x_n\}$ is given by (1.5). Then $\lim_{n\to\infty}\|x_n - T_jx_n\| = 0$ for each $j = 1,2,\ldots,r$.

In order to prove Theorem 2.1, we first prove the following lemmas.

Lemma 2.2. *Let C be a nonempty convex subset of a Banach space E. Let $T_j : C \to C$ be a uniformly L-Lipschitzian mapping for each $j = 1,2,\ldots,r$, and let $\{x_n\}$ be as in (1.5).*

Set $e_{n(j)} := \|x_n - T_j^n U_{n(j-1)} x_n\|$ *for all* $n, j \in \mathbb{N}$. *Then for all* $n \geq 2$,

$$\begin{aligned} ||x_n - T_1 x_n|| &\leq e_{n(1)} + (L^2 + L)e_{n-1(r)} + Le_{n-1(1)} + (L^2 + L)c_{n-1(r)}||u_{n-1(r)} - x_{n-1}||, \\ ||x_n - T_j x_n|| &\leq e_{n(j)} + (L^2 + L)e_{n-1(r)} + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + Le_{n-1(j)} \\ &\quad + (L^2 + L)c_{n-1(r)}||u_{n-1(r)} - x_{n-1}|| + L^2 c_{n(j-1)}||u_{n(j-1)} - x_n|| \\ &\quad + L^2 c_{n-1(j-1)}||x_{n-1} - u_{n-1(j-1)}||, \end{aligned} \tag{2.1}$$

for each $j = 2, 3, \ldots, r$.

Proof. Observe that for $j = 2, 3, \ldots, r$ we have

$$\begin{aligned} &||U_{n(j-1)}x_n - U_{n-1(j-1)}x_{n-1}|| \\ &\quad = ||(a_{n(j-1)}x_n + b_{n(j-1)}T_{j-1}^n U_{n(j-2)}x_n + c_{n(j-1)}u_{n(j-1)}) \\ &\qquad - (a_{n-1(j-1)}x_{n-1} + b_{n-1(j-1)}T_{j-1}^{n-1}U_{n-1(j-2)}x_{n-1} \\ &\qquad + c_{n-1(j-1)}u_{n-1(j-1)})|| \\ &\quad = ||(x_n - x_{n-1}) + b_{n(j-1)}(T_{j-1}^n U_{n(j-2)}x_n - x_n) \\ &\qquad + c_{n(j-1)}(u_{n(j-1)} - x_n) + b_{n-1(j-1)}(x_{n-1} - T_{j-1}^{n-1}U_{n-1(j-2)}x_{n-1}) \\ &\qquad + c_{n-1(j-1)}(x_{n-1} - u_{n-1(j-1)})|| \\ &\quad \leq ||x_n - x_{n-1}|| + e_{n(j-1)} + e_{n-1(j-1)} + c_{n(j-1)}||u_{n(j-1)} - x_n|| \\ &\qquad + c_{n-1(j-1)}||x_{n-1} - u_{n-1(j-1)}||, \end{aligned} \tag{2.2}$$

$$\begin{aligned} ||x_n - x_{n-1}|| &= ||a_{n-1(r)}x_{n-1} + b_{n-1(r)}T_r^{n-1}U_{n-1(r-1)}x_{n-1} + c_{n-1(r)}u_{n-1(r)} - x_{n-1}|| \\ &\leq b_{n-1(r)}||T_r^{n-1}U_{n-1(r-1)}x_{n-1} - x_{n-1}|| + c_{n-1(r)}||u_{n-1(r)} - x_{n-1}|| \\ &\leq e_{n-1(r)} + c_{n-1(r)}||u_{n-1(r)} - x_{n-1}||. \end{aligned} \tag{2.3}$$

Therefore,

$$\begin{aligned} ||x_n - T_j x_n|| &\leq ||x_n - T_j^n U_{n(j-1)}x_n|| + ||T_j^n U_{n(j-1)}x_n - T_j x_n|| \\ &\leq e_{n(j)} + L||T_j^{n-1}U_{n(j-1)}x_n - x_n|| \\ &\leq e_{n(j)} + L||T_j^{n-1}U_{n(j-1)}x_n - T_j^{n-1}U_{n-1(j-1)}x_{n-1}|| \\ &\quad + L||T_j^{n-1}U_{n-1(j-1)}x_{n-1} - x_{n-1}|| + L||x_{n-1} - x_n|| \\ &\leq e_{n(j)} + L^2||U_{n(j-1)}x_n - U_{n-1(j-1)}x_{n-1}|| \\ &\quad + Le_{n-1(j)} + L||x_{n-1} - x_n||. \end{aligned} \tag{2.4}$$

Using (2.3) in (2.4) for $j = 1$ we have

$$\begin{aligned} ||x_n - T_1 x_n|| &\leq e_{n(1)} + (L^2 + L)||x_n - x_{n-1}|| + Le_{n-1(1)} \\ &\leq e_{n(1)} + (L^2 + L)e_{n-1(r)} + Le_{n-1(1)} \\ &\quad + (L^2 + L)c_{n-1(r)}||U_{n-1(r)} - x_{n-1}||. \end{aligned} \tag{2.5}$$

Using (2.2) and (2.3) in (2.4) for $j = 2,3,\dots,r$ we have

$$\begin{aligned}||x_n - T_jx_n|| &\le e_{n(j)} + (L^2+L)||x_n - x_{n-1}|| + L^2e_{n(j-1)} + L^2e_{n-1(j-1)} + Le_{n-1(j)}\\ &\quad + L^2c_{n(j-1)}||u_{n(j-1)} - x_n|| + L^2c_{n-1(j-1)}||x_{n-1} - u_{n-1(j-1)}||\\ &\le e_{n(j)} + (L^2+L)e_{n-1(r)} + L^2e_{n(j-1)} + L^2e_{n-1(j-1)} + Le_{n-1(j)}\\ &\quad + (L^2+L)c_{n-1(r)}||u_{n-1(r)} - x_{n-1}|| + L^2c_{n(j-1)}||u_{n(j-1)} - x_n||\\ &\quad + L^2c_{n-1(j-1)}||x_{n-1} - u_{n-1(j-1)}||.\end{aligned} \tag{2.6}$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let C be a nonempty convex subset of a Banach space E. Let $\{T_1,T_2,\dots,T_r\}$, $\{u_{n(j)}\}$, and $\{x_n\}$ be as in Theorem 2.1 and let $\{a_{n(j)}\}$, $\{b_{n(j)}\}$, and $\{c_{n(j)}\}$ satisfy conditions (i) and (ii) of Theorem 2.1. Then $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in \cap_{j=1}^r F(T_j)$.*

Proof. Let $x^* \in \cap_{j=1}^r F(T_j)$. Since $\{u_{n(j)}\}_{n=1}^\infty$ and $\{k_n\}_{n=1}^\infty$ are bounded, there exists a constant $N > 0$ such that $\sup_{n\in\mathbb{N}}\{\|u_{n(j)} - x^*\| : j = 1,2,\dots,r\} \le N$ and $\sup_{n\in\mathbb{N}}\{1 + k_n + \cdots + k_n^{r-1}\} \le N$. Then, we have

$$\begin{aligned}||x_{n+1} - x^*|| &= ||a_{n(r)}x_n + b_{n(r)}T_r^nU_{n(r-1)}x_n + c_{n(r)}u_{n(r)} - x^*||\\ &\le a_{n(r)}||x_n - x^*|| + b_{n(r)}||T_r^nU_{n(r-1)}x_n - x^*|| + c_{n(r)}||u_{n(r)} - x^*||\\ &\le a_{n(r)}||x_n - x^*|| + b_{n(r)}k_n||U_{n(r-1)}x_n - x^*|| + Nc_{n(r)}\\ &= a_{n(r)}||x_n - x^*|| + b_{n(r)}k_n\\ &\quad \times ||a_{n(r-1)}(x_n - x^*) + b_{n(r-1)}(T_{r-1}^nU_{n(r-2)}x_n - x^*)\\ &\qquad + c_{n(r-1)}(u_{n(r-1)} - x^*)|| + Nc_{n(r)}\\ &\le [1 - b_{n(r)} + (1 - b_{n(r-1)})b_{n(r)}k_n]||x_n - x^*||\\ &\quad + b_{n(r)}b_{n(r-1)}k_n^2||U_{n(r-2)}x_n - x^*|| + Nc_{n(r)} + N^2c_{n(r-1)}\\ &\;\;\vdots\\ &\le \Big[1 - b_{n(r)} + (1 - b_{n(r-1)})b_{n(r)}k_n\\ &\qquad + \cdots + (1 - b_{n(1)})b_{n(r)}b_{n(r-1)}\cdots b_{n(2)}k_n^{r-1} + b_{n(r)}b_{n(r-1)}\cdots b_{n(1)}k_n^r\Big]\\ &\quad \cdot ||x_n - x^*|| + N(c_{n(r)} + Nc_{n(r-1)} + \cdots + Nc_{n(1)})\\ &= \Big[1 + b_{n(r)}(k_n - 1) + b_{n(r)}b_{n(r-1)}k_n(k_n - 1)\\ &\qquad + \cdots + b_{n(r)}b_{n(r-1)}\cdots b_{n(1)}(k_n^{r-1})(k_n - 1)\Big]||x_n - x^*|| + \psi_n\\ &\le [1 + (k_n - 1)(1 + k_n + \cdots + k_n^{r-1})]||x_n - x^*|| + \psi_n\\ &\le [1 + (k_n - 1)N]||x_n - x^*|| + \psi_n\\ &= (1 + \varphi_n)||x_n - x^*|| + \psi_n,\end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$, where $\varphi_n := (k_n - 1)N$ and $\psi_n := N(c_{n(r)} + Nc_{n(r-1)} + \cdots + Nc_{n(1)})$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} c_{n(j)} < \infty$ for each $j = 1,2,\dots,r$, we have $\sum_{n=1}^{\infty}\varphi_n < \infty$ and $\sum_{n=1}^{\infty}\psi_n < \infty$. Thus, $\lim_{n\to\infty}\|x_n - x^*\|$ exists by Lemma 1.1. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Under the hypotheses of Lemma 2.3, if E is a uniformly convex Banach space, then there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$, and*

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{r}\left(\prod_{l=j}^{r}\alpha_{n(l)}\right)(1-\alpha_{n(j)})g(\|x_n - T_j^n U_{n(j-1)}x_n\|)\right] < \infty, \tag{2.8}$$

where $\alpha_{n(j)} := b_{n(j)} + c_{n(j)}$ for all $n \in \mathbb{N}$ and each $j = 1,2,\dots,r$.

Proof. Let $x^* \in \cap_{j=1}^{r} F(T_j)$. Lemma 2.3 and the hypotheses of Lemma 2.4 imply that $\{x_n - x^*\}_{n=1}^{\infty}$, $\{u_{n(j)}\}_{n=1}^{\infty}$, and $\{k_n\}_{n=1}^{\infty}$ are bounded. Then, there exists a constant $d > 0$ such that

$$\cup_{j=1}^{r}\{T_j^n U_{n(j-1)}x_n - x^*\}_{n=1}^{\infty} \cup \{x_n - x^*\}_{n=1}^{\infty} \subseteq B_d. \tag{2.9}$$

By Theorem 1.3, there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(0) = 0$, and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|) \tag{2.10}$$

for all $x, y \in B_d$ and $\lambda \in [0,1]$. By inequality (2.10) we obtain the following estimate: for some constant M, we have

$$\begin{aligned}
\|U_{n(j)}x_n - x^*\|^2 &= \|(1-\alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)}x_n - x^*) \\
&\qquad - c_{n(j)}(T_j^n U_{n(j-1)}x_n - u_{n(j)})\|^2 \\
&\le \Big(\|(1-\alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)}x_n - x^*)\| \\
&\qquad + c_{n(j)}\|(T_j^n U_{n(j-1)}x_n - u_{n(j)})\|\Big)^2 \\
&\le \|(1-\alpha_{n(j)})(x_n - x^*) + \alpha_{n(j)}(T_j^n U_{n(j-1)}x_n - x^*)\|^2 + c_{n(j)}M \\
&\le (1-\alpha_{n(j)})\|x_n - x^*\|^2 + \alpha_{n(j)}\|T_j^n U_{n(j-1)}x_n - x^*\|^2 \\
&\qquad - \alpha_{n(j)}(1-\alpha_{n(j)})g(\|x_n - T_j^n U_{n(j-1)}x_n\|) + c_{n(j)}M \\
&\le (1-\alpha_{n(j)})\|x_n - x^*\|^2 + \alpha_{n(j)}k_n^2\|U_{n(j-1)}x_n - x^*\|^2 \\
&\qquad - \alpha_{n(j)}(1-\alpha_{n(j)})g(\|x_n - T_j^n U_{n(j-1)}x_n\|) + c_{n(j)}M,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1-\alpha_{n(r)})(x_n - x^*) + \alpha_{n(r)}(T_r^n U_{n(r-1)}x_n - x^*) \\
&\qquad - c_{n(r)}(T_r^n U_{n(r-1)}x_n - u_{n(r)})\|^2 \\
&\le (1-\alpha_{n(r)})\|x_n - x^*\|^2 + \alpha_{n(r)}k_n^2\|U_{n(r-1)}x_n - x^*\|^2 \\
&\qquad - \alpha_{n(r)}(1-\alpha_{n(r)})g(\|x_n - T_r^n U_{n(r-1)}x_n\|) + c_{n(r)}M.
\end{aligned} \tag{2.12}$$

By a repeated application of inequality (2.11) in (2.12), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le{}& \|x_n-x^*\|^2\\ &+\alpha_{n(r)}(k_n^2-1)\big(1+\alpha_{n(r-1)}k_n^2+\cdots\\ &\qquad+\alpha_{n(r-1)}\alpha_{n(r-2)}\cdots\alpha_{n(1)}k_n^{2(r-1)}\big)\|x_n-x^*\|^2\\ &-\sum_{j=1}^{r}\left(\prod_{l=j}^{r}\alpha_{n(l)}\right)(1-\alpha_{n(j)})g(\|x_n-T_j^nU_{n(j-1)}x_n\|)\\ &+\big(c_{n(r)}+k_n^2c_{n(r-1)}+\cdots+k_n^{2(r-1)}c_{n(1)}\big)M.\end{aligned}\tag{2.13}$$

Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$, hence $\lim_{n\to\infty}k_n=1$, we may assume that $k_n\le L$ for all $n\in\mathbb{N}$ and some constant L. Let $N=\max_{1\le j\le r}\{L^{2j}\}\ge 1$. Then

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le{}& \|x_n-x^*\|^2+(k_n-1)(N+1)rNd^2+MN\sum_{j=1}^{r}c_{n(j)}\\ &-\sum_{j=1}^{r}\left(\prod_{l=j}^{r}\alpha_{n(l)}\right)(1-\alpha_{n(j)})g(\|x_n-T_j^nU_{n(j-1)}x_n\|)\end{aligned}\tag{2.14}$$

for all $n\in\mathbb{N}$. Transposing and summing from 1 to m we have

$$\begin{aligned}&\sum_{n=1}^{m}\left[\sum_{j=1}^{r}\left(\prod_{l=j}^{r}\alpha_{n(l)}\right)(1-\alpha_{n(j)})g(\|x_n-T_j^nU_{n(j-1)}x_n\|)\right]\\ &\quad\le\|x_1-x^*\|^2+(N+1)rNd^2\sum_{n=1}^{m}(k_n-1)+MN\sum_{n=1}^{m}\sum_{j=1}^{r}c_{n(j)}.\end{aligned}\tag{2.15}$$

Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$ and $\sum_{n=1}^{\infty}c_{n(j)}<\infty$ for each $j=1,2,\dots,r$, it follows that

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{r}\left(\prod_{l=j}^{r}\alpha_{n(l)}\right)(1-\alpha_{n(j)})g(\|x_n-T_j^nU_{n(j-1)}x_n\|)\right]<\infty.\tag{2.16}$$

This completes the proof of Lemma 2.4. □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.4 and condition (iii), we have

$$(1-b)\sum_{n=1}^{\infty}\sum_{j=1}^{r}a^{r-j+1}g(\|x_n-T_j^nU_{n(j-1)}x_n\|)<\infty.\tag{2.17}$$

Thus,

$$\sum_{j=1}^{r}g(\|x_n-T_j^nU_{n(j-1)}x_n\|)\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\tag{2.18}$$

Since g is a continuous and strictly increasing function with $g(0)=0$, we have $\lim_{n\to\infty}\|x_n-T_j^nU_{n(j-1)}x_n\|=0$ for each $j=1,2,\dots,r$. Since $\{x_n-x^*\}$ and $\{u_{n(j)}\}$ are bounded. So we have

$$\sup_{n\in\mathbb{N}}\{\|x_n-u_{n(j)}\|:j=1,2,\dots,r\}\le D\tag{2.19}$$

for some constant $D > 0$. Let $e_{n(j)} = \|x_n - T_j^n U_{n(j-1)} x_n\|$ and L be as in the proof of Lemma 2.4. Then, by Lemma 2.2, we have

$$\begin{aligned} \|x_n - T_1 x_n\| &\le e_{n(1)} + (L^2+L)e_{n-1(r)} + Le_{n-1(1)} + (L^2+L)c_{n-1(r)}D \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ \|x_n - T_j x_n\| &\le e_{n(j)} + (L^2+L)e_{n-1(r)} + L^2 e_{n(j-1)} + L^2 e_{n-1(j-1)} + Le_{n-1(j)} \\ &\quad + (L^2+L)c_{n-1(r)}D + L^2 c_{n(j-1)}D + L^2 c_{n-1(j-1)}D \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{2.20}$$

for each $j = 2,3,\dots,r$. This completes the proof of Theorem 2.1. □

Theorem 2.5. *Under the hypotheses of Theorem 2.1, if E is a uniformly convex Banach space satisfying Opial's condition, then $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, \dots, T_r$.*

Proof. Let $\omega_w(\{x_n\})$ be the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in C. By Lemma 1.4 and Theorem 2.1, $\omega_w(\{x_n\})$ is contained in $\cap_{j=1}^r F(T_j)$.

The remainder of the proof is similar to that of [9, Theorem 2.1], so the details are omitted. □

Remark 2.6. Theorem 2.5 generalizes [9, Theorem 2.1].

Theorem 2.7. *Under the hypotheses of Theorem 2.1. Suppose that T_1^m is compact for some $m \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \dots, T_r$.*

Proof. As in the proof of [9, Theorem 2.2] by using Theorem 2.1 and Lemma 2.3, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $\lim_{i\to\infty} x_{n_i} = p$. Thus, by Theorem 2.1, we obtain that $T_j p = p$ for each $j = 1,2,\dots,r$. Hence, $p \in \cap_{j=1}^r F(T_j)$ and it follows from Lemma 2.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Therefore, we conclude that $\lim_{n\to\infty} \|x_n - p\| = 0$, completing the proof of Theorem 2.7. □

Remark 2.8. Theorem 2.7 generalizes [9, Theorem 2.2] and [6, Theorems 2 and 3].

Lemma 2.9. *Let K be a compact convex subset of a normed space E. Suppose that $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$. Then*

$$d(\alpha x + \beta y + \gamma z, K) \le \alpha d(x,K) + \beta d(y,K) + \gamma d(z,K) \tag{2.21}$$

for all $x, y, z \in E$ where $d(x,K) := \inf\{\|x - p\| : p \in K\}$.

Proof. Let $x, y, z \in E$. Since K is compact, we have $d(x,p_1) = d(x,K)$, $d(y,p_2) = d(y,K)$, and $d(z,p_3) = d(z,K)$ for some p_1, p_2, $p_3 \in K$. Since K is convex so that $\alpha p_1 + \beta p_2 + \gamma p_3 \in K$. Therefore,

$$\begin{aligned} d(\alpha x + \beta y + \gamma z, K) &\le \|(\alpha x + \beta y + \gamma z) - (\alpha p_1 + \beta p_2 + \gamma p_3)\| \\ &\le \alpha\|x - p_1\| + \beta\|y - p_2\| + \gamma\|z - p_3\| \\ &= \alpha d(x,K) + \beta d(y,K) + \gamma d(z,K). \end{aligned} \tag{2.22}$$

This completes the proof of Lemma 2.9. □

Theorem 2.10. *Under the hypotheses of Theorem 2.1. Suppose that there exists a nonempty compact convex subset K of E and some $\alpha \in (0,1)$ such that $d(T_j x, K) \leq \alpha d(x,K)$ for all $x \in C$ and each $j = 1,2,\dots,r$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \dots, T_r$.*

Proof. For $n \in \mathbb{N}$ and $x \in C$ we have $d(T_j^n x, K) \leq \alpha^n d(x,K)$ for each $j = 1,2,\dots,r$. Since $\{u_{n(j)}\}_{n=1}^{\infty}$ is bounded for each $j = 1,2,\dots,r$ and K is compact. Thus, there exists a constant $D > 0$ such that

$$\sup_{n\in\mathbb{N}} \{d(u_{n(j)}, K) : j = 1,2,\dots,r\} \leq D. \tag{2.23}$$

Then, by Lemma 2.9, we have

$$\begin{aligned}
d(x_{n+1}, K) &= d(a_{n(r)} x_n + b_{n(r)} T_r^n U_{n(r-1)} x_n + c_{n(r)} u_{n(r)}, K) \\
&\leq a_{n(r)} d(x_n, K) + b_{n(r)} d(T_r^n U_{n(r-1)} x_n, K) + c_{n(r)} d(u_{n(r)}, K) \\
&\leq a_{n(r)} d(x_n, K) + b_{n(r)} \alpha^n d(U_{n(r-1)} x_n, K) + c_{n(r)} D \\
&\leq (1 - b_{n(r)}) d(x_n, K) + b_{n(r)} \alpha^n d(a_{n(r-1)} x_n + b_{n(r-1)} T_{r-1}^n U_{n(r-2)} x_n \\
&\quad + c_{n(r-1)} u_{n(r-1)}, K) + c_{n(r)} D \\
&\leq [1 - b_{n(r)} + (1 - b_{n(r-1)}) b_{n(r)} \alpha^n] d(x_n, K) \\
&\quad + b_{n(r)} b_{n(r-1)} \alpha^{2n} d(U_{n(r-2)} x_n, K) + (c_{n(r-1)} + c_{n(r)}) D \\
&\ \ \vdots \\
&\leq \Big[1 - b_{n(r)} (1 - \alpha^n)(1 + b_{n(r-1)} \alpha^n + \cdots + b_{n(r-1)} b_{n(r-2)} \cdots b_{n(1)} \alpha^{(r-1)n})\Big] \\
&\quad d(x_n, K) + (c_{n(1)} + c_{n(2)} + \cdots + c_{n(r)}) D \\
&\leq [1 - a(1 - \alpha^n)(1 + a\alpha^n + \cdots + a^{r-1} \alpha^{(r-1)n})] d(x_n, K) + D \sum_{j=1}^{r} c_{n(j)}.
\end{aligned} \tag{2.24}$$

Let $\delta_n = a(1-\alpha^n)(1 + a\alpha^n + \cdots + a^{r-1}\alpha^{(r-1)n})$. Since $\lim_{n\to\infty} \delta_n = a$ and $0 < a < 1$, then there exists a real number $N_0 \geq 1$ such that $\delta_n < 1$ for all $n \geq N_0$. Since $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} \sum_{j=1}^{r} c_{n(j)} < \infty$, then by Lemma 1.2, we have $\lim_{n\to\infty} d(x_n, K) = 0$. Since K is compact, this is easily seen to imply that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ such that $\lim_{i\to\infty} x_{n_i} = p$. The rest of the proof is identical to the related part of the proof of Theorem 2.7. □

Remark 2.11. Theorem 2.10 generalizes [9, Theorem 2.4].

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