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A NOTE ON NONFRAGMENTABILITY OF BANACH SPACES

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ABSTRACT. We use Kenderov-Moors characterization of fragmentability to show that if a compact Hausdorff space X with the tree-completeness property contains a disjoint sequences of clopen sets, then $(C(X), \text{weak})$ is not fragmented by any metric which is stronger than weak topology. In particular, $C(X)$ does not admit any equivalent locally uniformly convex renorming.

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1. Introduction. Let (X,τ) be a topological space and ρ a metric on X. Given $\epsilon > 0$, a nonempty subset A of X is said to be *fragmented* by ρ down to ϵ if each nonempty subset of A has a nonempty τ-relatively open subset of A with ρ-diameter less than ϵ. The set A is said to be fragmented by ρ if A is fragmented by ρ down to ϵ for each $\epsilon > 0$. The set A is said to be *sigma-fragmented by* ρ [7] if for each $\epsilon > 0$, A can be expressed as $A = \bigcup_{n=1}^{\infty} A_{n,\epsilon}$ with each $A_{n,\epsilon}$ fragmented by ρ down to ϵ.

The notion of fragmentability was originally introduced in [11] as an abstraction of phenomena often encountered, for example, in Banach spaces with the Radon-Nikodym property, in weakly compact subsets of Banach spaces and in the dual of Banach spaces. The notion of σ-fragmentability appeared in [10] in order to extend the study of compact fragmented space to noncompact spaces. It turns out that the question of whether a given Banach space with weak topology is sigma-fragmented by the norm is closely connected with the question of the existence of an equivalent Kadec and locally uniformly convex norm. The reader may refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] for some application of fragmentability and its variants in other topics of Banach spaces.

Kenderov and Moors [13, 14] used the following topological game to characterize fragmentability and sigma-fragmentability of a topological space X.

Two players Σ and Ω alternatively select subsets of X. The player Σ usually starts the game by choosing some nonempty subset A_1 of X, then the Ω-player chooses some nonempty relatively open subset A_1, say B_1, then Σ will choose a nonempty set $A_2 \subset B_1$ and in turn, Ω picks up some nonempty relatively open subset B_2 of A_2. By continuing this procedure, the two players generate a sequence of sets

$$A_1 \supset B_1 \supset \cdots \supset A_n \supset B_n \supset \cdots \tag{1.1}$$

which is called a play and is denoted by $p = (A_i, B_i)_{i=1}^{\infty}$. If

$$p_k = (A_1, B_1, \ldots, A_k) \quad (1 \le k \le n) \tag{1.2}$$

are the first "n" move of some play (of the game), then we call p_k a *partial play* of the game. The player Ω is said to have won the play if $\bigcap_{i=1}^{\infty} A_i$ contains at most one point. Otherwise the player Σ is said to have won this play. Under the term *strategy* for the player Ω, we understand a mapping ω which assigns to every partial play p_n a nonempty relatively open subset $B_n = \omega(p_n)$ of A_n. The play $(A_i, B_i)_{i=1}^{\infty}$ is called an *ω-play* if $B_i = \omega(p_i)$ for every $i \geq 1$. Similarly, the partial play p_n is called a *partial ω-play*, if $B_i = \omega(p_i)$ for each $i < n$. The map ω is called a *winning strategy* for the player Ω if he/she wins every ω-play. If the space X is fragmentable by a metric $d(\cdot,\cdot)$, then Ω has an obvious winning strategy ω. Indeed, to each partial play p_n this strategy puts into correspondence some nonempty subset $B_n \subset A_n$ which is relatively open in A_n and has d-diameter less than $1/n$. Clearly, the set $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i$ has at most one point because it has d-diameter 0. It turns out that the existence of a winning strategy for player Ω characterizes fragmentability.

Theorem 1.1 (see [13]). *The topological space X is fragmentable if and only if the player Ω has a winning strategy.*

By Theorem 1.1, it was shown in [15] that X/c_0, where X is the Haydon-Zizler subspace of ℓ^∞ [5] is not fragmented by any metric. According to a result of Ribarska [18], if a Banach space admits an equivalent strictly convex renorming, then it is fragmented by a metric. It follows that X/c_0 does not admit strictly convex renorming. This could be considered as an extension of [1].

Although ℓ^∞ taken with its weak topology is not sigma-fragmented by the norm, it is fragmented by a lower semi-continuous metric (see [9, Example 3.2]). However, in [14], it is shown that fragmentability and sigma-fragmentability in a Banach space may be related to each other in the following way.

Theorem 1.2 (see [14, Theorems 1.3, 1.4, and 2.1]). *For a Banach space X the following are equivalent:*

(i) *$(X$, weak$)$ is sigma-fragmented by a metric which is stronger than the weak topology;*
(ii) *$(X$, weak$)$ is fragmented by a metric which is stronger than the weak topology;*
(iii) *there exists a strategy ω for the player Ω such that, for every ω-play $p = (A_i, B_i)_i$ either $\bigcap_{i\geq 1} B_i = \emptyset$ or $\lim_{i\to\infty}$ norm-$\operatorname{diam}(B_i) = 0$.*

It is known that whenever X is compact and extremely disconnected, then $C(X)$ contains an isometric copy of ℓ^∞ (see [2, page 18]), therefore it is not sigma-fragmented by the norm. However, there exists a compact Hausdorff space X (with the tree completeness property) such that $C(X)$ does not contain a copy of ℓ^∞ (see [4]). It is natural to ask if such a space is sigma-fragmented by the norm. The above result enable us to give an answer to this question. More precisely, thanks to Theorem 1.2, we will show that if a compact Hausdorff space X with the tree-completeness property has a sequence of disjoint clopen sets, then $(C(X)$, weak$)$ is not (sigma) fragmented by any metric which is stronger than the weak topology. It follows that $C(X)$ does not admit any equivalent locally uniformly convex norm.

2. Results.

Let $T = \bigcup_{k=0}^{\infty}\{0,1\}^k$. The elements of T, are finite (possibly empty) strings of 0's and 1's. The empty string ( ) is the unique string of length 0; more generally, the *length* $|t|$ of a string t is n if $t \in \{0,1\}^n$. The *tree-order* is defined by $s \prec t$ if $|s| < |t|$ and $t(m) = s(m)$ for $m \le |s|$. Each $t \in T$ has exactly two immediate successors, that is, $t0$ and $t1$.

A topological space X is said to have *the tree-completeness property* if whenever $\{V_t\}_{t\in T}$ is a sequence of disjoint clopen sets in X there exists some $b \in \{0,1\}^{N^*}$, $N^* = N \cup \{0\}$, such that $\overline{\bigcup_{n\in N^*} V_{b|n}}$ is open. Evidently, every infinite extremally disconnected space [3] has the tree-completeness property. However, as it was mentioned in Section 1, there exists a compact Hausdorff space with the tree-completeness property which is not extremally disconnected.

Definition 2.1. A subset Y of a compact Hausdorff space X is C^*-embedded [3] in X if every function in $C(Y)$ can be extended to a function in $C(X)$.

Lemma 2.2. *Let $\{N_t\}_{t\in T}$ be a sequence of infinite subsets of N, such that*

(i) *$N_t \subset N_s$, whenever $s \prec t$.*

(ii) *$N_t \cap N_s = \varnothing$, if t and s are not comparable.*

Let $\{V_n\}_{n\in N^*}$ be a sequence of clopen subsets of a compact Hausdorff space X, such that $\overline{\bigcup_{k\in N_t} V_k}$ is open for each $t \in T$. If X has the tree-completeness property, then there exists some $b \in \{0,1\}^{N^*}$, such that $\bigcup_{n=0}^{\infty}(X \setminus \overline{\cup_{k\in N_{b|n}} V_k})$ is C^*-embedded.

Proof. Let

$$Z_{(\,)} = X \setminus \overline{\cup_{k\in N_{(\,)}} V_k}, \qquad Z_{ti} = (X \setminus \overline{\cup_{k\in N_{ti}} V_k}) \setminus \bigcup_{s\preceq t} Z_s, \tag{2.1}$$

for $i = 0,1$ and $t \in T$. Then $\{Z_t\}_{t\in T}$ is a sequence of disjoint clopen subsets of X. By the tree-completeness property of X, there exists some $b \in \{0,1\}^{N^*}$, such that

$$\bigcup_{n\in N^*} Z_{b|n} = \bigcup_{n\in N^*} (X \setminus \overline{\cup_{k\in N_{b|n}} V_k}) \tag{2.2}$$

is clopen in X, thus it is C^*-embedded. □

Lemma 2.3. *Let $\{V_n\}_{n\in N}$ be an infinite disjoint sequence of clopen subsets of a compact Hausdorff space X and $\mu \in C(X)^*$, where X has the tree-completeness property. Then there exists an infinite set $N_1 \subset N$, such that $\overline{\cup_{n\in N_1} V_n}$ is clopen subset of X and $|\mu(f)| < \epsilon$, whenever* $\operatorname{supp}(f) \subset \overline{\cup_{n\in N_1} V_n}$ *and $\|f\| \le 2$.*

Proof. Suppose that $2\|\mu\| < n\epsilon$. Note that for every infinite subset M of N, there exists some infinite subset M_1 of M such that $\overline{\cup_{n\in M_1} V_n}$ is clopen.

If the lemma were not true, we can find infinite disjoint subsets $M_1,\dots,M_n$ of N and continuous functions $f_1,\dots,f_n$ such that

$$\operatorname{supp}(f_i) \subset \overline{\cup_{n\in M_i} V_n}(\text{clopen}), \quad \|f_i\| \le 2,\ \mu(f_i) \ge \epsilon. \tag{2.3}$$

Put $f = \sum_{i=1}^n f_i$, since f_i's have disjoint support, we have $\|f\| \le 2$, but $\mu(f) = \sum_{i=1}^n \mu(f_i) \ge n\epsilon$. This is a contradiction. □

Theorem 2.4. *Let X be a compact Hausdorff space with the tree-completeness property. If X contains a disjoint sequence of clopen sets. Then $(C(X), \text{weak})$ is not (sigma) fragmented by any metric which is stronger than weak topology.*

Proof. By Theorem 1.2, it is enough to show that for each strategy ω for the player Ω there exists an ω-play $p = (A_i, B_i)_i$ such that, $\bigcap_{i\geq 1} B_i \neq \emptyset$ and $\lim_{i\to\infty}$ norm-diam$(B_i) > 0$. Fix a strategy ω for the player Ω. By induction on $|t|$, $t \in T$, we will construct partial ω-plays $p_t = (A_{(\,)}, B_{(\,)}, A_{t|1}, \dots, A_t)$. Then, we will show that there is some $b \in \{0,1\}^{N^*}$, such that the ω-play $p_b = (A_{(\,)}, B_{(\,)}, A_{b|1}, \dots)$ has the required properties.

Let $\{V_n\}_{n\in N}$ be an infinite disjoint sequence of nonempty clopen subsets of X. Let $N_{(\,)}$ be an infinite subset of N such $\overline{\bigcup_{n\in N_{(\,)}} V_n}$ is a clopen subset of X. For some $f_{(\,)}$ in the unit ball of $C(X)$, we define

$$A_{(\,)} = \left\{ f : \|f\| \leq 1,\ f(x) = f_{(\,)}(x) \text{ for } x \in X \setminus \overline{\bigcup_{n\in N_{(\,)}} V_n} \right\} \tag{2.4}$$

as the first choice of the player Σ. Therefore, we have the partial ω-play $p_{(\,)} = (A_{(\,)})$, clearly norm-diam$(A_{(\,)}) = 1$. Suppose that for every t with $|t| \leq n$, the partial ω-play $p_t = (A_{(\,)}, B_{(\,)}, A_{t|1}, B_{t|1}, \dots, A_t)$ has already been defined. Let $B_t = \omega(p_t)$ be the relatively open subset of A_t, chosen by the player Ω according to his/her strategy as the answer to this movement. Let $f_t' \in B_t$, since B_t is a relatively open subset of A_t, there are linear functionals $\mu_1^t, \dots, \mu_{K_t}^t$ on $C(X)$ and $\epsilon_t > 0$, such that

$$\{f \in A_t : \|f\| \leq 1,\ |\mu_i^t(f - f_t')| < \epsilon_t,\ 1 \leq i \leq K_t\} \subset B_t. \tag{2.5}$$

Applying Lemma 2.3, we can find an infinite subset N_t' of N_t, such that $\overline{\bigcup_{n\in N_t'} V_n}$ is clopen and

$$|\mu_i^t(f)| < \epsilon_t \quad \text{whenever } \operatorname{supp}(f) \subset \overline{\bigcup_{n\in N_t'} V_n},\ \|f\| \leq 2 \text{ for } 1 \leq i \leq K_t. \tag{2.6}$$

Suppose that N_{t0} and N_{t1} are two disjoint infinite subset of N_t', such that each $\overline{\bigcup_{n\in N_{ti}} V_n}$ is clopen, $i = 0,1$. Let $f_{ti} = f_t' \cdot \chi_{X\setminus \overline{\bigcup_{n\in N_{ti}} V_n}}$ and define

$$A_{ti} = \left\{ f \in A_t : f(x) = f_{ti}(x) \text{ for } x \in X \setminus \overline{\bigcup_{n\in N_{ti}} V_n} \right\} \quad (i = 0,1). \tag{2.7}$$

Then A_{t0} and A_{t1} are subsets of B_t with norm diameter 1 and we have the partial ω-plays

$$p_{ti} = (A_{(\,)}, B_{(\,)}, A_{t|1}, B_{t|1}, \dots, A_t, B_t, A_{ti}) \quad (i = 0,1). \tag{2.8}$$

Thus, by induction on $|t|$, we proved that, there are partial ω-plays

$$p_t = (A_{(\,)}, B_{(\,)}, \dots, A_t), \quad (t \in T), \tag{2.9}$$

such that the following conditions hold:

(i) A_t is of the form

$$\left\{ f : \|f\| \leq 1,\ f(x) = f_t(x) \text{ for } x \in X \setminus \overline{\bigcup_{n\in N_t} V_n} \right\}, \tag{2.10}$$

(ii) for each N_t, $\overline{\bigcup_{n\in N_t} V_n}$ is clopen,

(iii) $N_t \subset N_s$, when $s \prec t$,
(iv) $N_t \cap N_s = \varnothing$, when s and t are not comparable,
(v) norm-diam$(A_t) = 1$ for each $t \in T$,
(vi) $f_t(x) = f_{ti}(x)$ for $x \in X \setminus \overline{\bigcup_{n\in N_t} V_n}$ and $i = 0, 1$.

Applying Lemma 2.2, we can find some $b \in \{0,1\}^{N^*}$, such that every continuous function on $\bigcup_{n\in N^*}(X \setminus \overline{\bigcup_{k\in N_{b|n}} V_k})$ has a continuous extension on X. By (vi), the function $f_b^*(x) = \lim_{n\to\infty} f_{b|n}(x)$ is continuous on $\bigcup_{n\in N^*}(X \setminus \overline{\bigcup_{k\in N_{b|n}} V_k})$, thus it has a continuous extension f_b on X without increasing norm. Clearly $f_b \in \bigcap_{n\in N^*} A_{b|n}$. Thus $\bigcap A_{b|n} \neq \varnothing$ and $\lim_{n\to\infty}$ norm-diam$(A_{b|n}) = 1$, that is, the ω-play $p_b = (A_{(\,)}, B_{(\,)}, A_{b|1}, B_{b|1}, \dots)$ does not satisfy Theorem 1.2(iii). This proves the theorem. □

Corollary 2.5. *If a compact Hausdorff space X with the tree-completeness property has an infinite sequence of clopen sets, then $C(X)$ does not admit any equivalent locally uniformly convex norm.*

Proof. It is known that if $(C(X)$, weak) admits an equivalent locally uniformly convex norm then it is norm-fragmented (see [7, Theorem 4.2]). Thus the result follows from Theorem 2.4. □

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