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GENERALIZED PERIODIC AND GENERALIZED BOOLEAN RINGS

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Abstract. We prove that a generalized periodic, as well as a generalized Boolean, ring is either commutative or periodic. We also prove that a generalized Boolean ring with central idempotents must be nil or commutative. We further consider conditions which imply the commutativity of a generalized periodic, or a generalized Boolean, ring.

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Throughout, R denotes a ring, N the set of nilpotents, C the center, and E the set of idempotents of R. A ring R is called *periodic* if for every x in R, there exist distinct positive integers m,n such that $x^m = x^n$. We now formally state the definitions of a generalized periodic ring and a generalized Boolean ring.

Definition 1. A ring R is called *generalized periodic* if for every x in R such that $x \notin (N \cup C)$, we have $x^n - x^m \in (N \cap C)$, for some positive integers m,n *of opposite parity*.

Definition 2. A ring R is called *generalized Boolean* if for every x in R such that $x \notin (N \cup C)$, there exists an *even* positive integer n such that $x - x^n \in (N \cap C)$.

Theorem 3. *If R is a generalized periodic ring, then R is either commutative or periodic.*

Proof. Let N and C denote the set of nilpotents and the center of R, respectively. We distinguish three cases.

Case 1 ($N \subseteq C$). Then $x \notin C$ implies $x \notin (N \cup C)$, and hence there exist distinct positive integers m,n such that $x^m - x^n \in N$, with $n > m$. Suppose $(x^m - x^n)^k = 0$. Then, as is readily verified,

$$(x - x^{n-m+1})^k x^{k(m-1)} = 0, \tag{1}$$

which, in turn, implies that

$$\begin{aligned}(x - x^{n-m+1})^{km} &= (x - x^{n-m+1})^k x^{k(m-1)} g(x) \\ &= 0,\end{aligned} \tag{2}$$

where

$$g(\lambda) \in \mathbb{Z}[\lambda]. \tag{3}$$

We have thus shown that

$$x - x^{n-m+1} \in N, \quad \forall x \notin C, \ (n-m+1 > 1). \tag{4}$$

Recall that, in our present case, we assumed that $N \subseteq C$, and hence by (4),

$$x - x^{n-m+1} \in C, \quad \forall x \notin C, \ (n - m + 1 > 1). \tag{5}$$

Since (5) is trivially satisfied if $x \in C$, we see that

$$x - x^{n(x)} \in C, \quad \text{for some } n(x) > 1, \text{ where } x \in R \text{ (arbitrary)}. \tag{6}$$

Therefore, R is commutative, by a well-known theorem of Herstein [3].

Case 2 ($C \subseteq N$)**.** Then $x \notin N$ implies $x \notin (N \cup C)$, and hence there exist distinct positive integers m, n such that $x^n - x^m \in N$, with $n > m$. Repeating the argument used to prove (4), we see that

$$x - x^{n-m+1} \in N, \quad \forall x \notin N, \ (n - m + 1 > 1). \tag{7}$$

Since (7) is trivially satisfied for all $x \in N$, we conclude that

$$x - x^{k(x)} \in N, \quad \text{for some } k(x) > 1, \text{ where } x \in R \text{ (arbitrary)}. \tag{8}$$

By a well-known theorem of Chacron [2], equation (8) implies that R is periodic.

Case 3 ($C \nsubseteq N$ and $N \nsubseteq C$)**.** In this case, let

$$z \in C \backslash N, \qquad u \in N \backslash C. \tag{9}$$

Equation (9) readily implies that $z + u \notin C$ and $z + u \notin N$, and hence (see Definition 1)

$$(z+u)^n - (z+u)^m \in N, \quad \text{for some integers } n > m \geq 1. \tag{10}$$

Since z commutes with the nilpotent element u, (10) implies that

$$z^n - z^m + u' \in N, \quad \text{where } u' \in N, \ u' \text{commutes with } z. \tag{11}$$

Hence $z^n - z^m \in N$, with $n > m \geq 1$. Now, a repetition of the argument used in the proof of (4) shows that

$$z - z^{n-m+1} \in N, \quad \forall z \in C \backslash N, \ (n - m + 1 > 1). \tag{12}$$

Trivially,

$$x - x^k \in N, \quad \forall x \in N, \ \forall k \in \mathbb{Z}^+. \tag{13}$$

Finally, if $x \notin (N \cup C)$, then

$$x^n - x^m \in N, \quad \text{for some integers } n > m \geq 1. \tag{14}$$

Again, repeating the argument used in the proof of (4), we see that

$$x - x^{n-m+1} \in N, \quad \forall x \notin (N \cup C), \ (n - m + 1 > 1). \tag{15}$$

Combining (12), (13), and (15), we conclude that

$$x - x^{k(x)} \in N, \quad \text{for some } k(x) > 1, \text{ where } x \in R \text{ (arbitrary)}. \tag{16}$$

Thus, by Chacron's theorem [2], R is periodic. This completes the proof. □

Corollary 4. *If R is a generalized Boolean ring, then R is either commutative or periodic.*

This follows at once, since a generalized Boolean ring is necessarily a generalized periodic ring (see Definitions 1 and 2).

Before proving the next theorem, we prove the following lemma.

Lemma 5. *Let R be a generalized periodic ring. If e is any nonzero central idempotent in R and $a \in N$, then $ea \in C$.*

Proof. The proof is by contradiction. Suppose the lemma is false, and let

$$\eta_0 \in N, \quad e\eta_0 \notin C. \tag{17}$$

Since $e \in C$ and $\eta_0 \in N$, therefore $e\eta_0$ is nilpotent. Let

$$(e\eta_0)^{\alpha} \in C, \quad \forall \alpha \geq \alpha_0, \ \alpha_0 \text{ minimal.} \tag{18}$$

Since $e\eta_0 \notin C$ (see (17)), therefore $\alpha_0 > 1$. Let $\eta = (e\eta_0)^{\alpha_0 - 1}$. Then,

$$\begin{gathered} \eta = (e\eta_0)^{\alpha_0-1} \in N, \quad \eta \notin C \text{ (by the minimality of } \alpha_0\text{)}, \\ \eta^k \in C, \quad \forall k \geq 2, \qquad e \in C, \quad e^2 = e \neq 0, \quad e \notin N. \end{gathered} \tag{19}$$

Equation (19) implies that $e + \eta \notin C$ and $e + \eta \notin N$, and hence (see Definition 1)

$$(e+\eta)^{m'} - (e+\eta)^{n'} \in C, \tag{20}$$

where m', n' are of *opposite* parity. Combining (20) and (19), we see that (keep in mind that $e\eta = \eta$; see (19))

$$(m' - n')e\eta \in C, \tag{21}$$

where $m' - n'$ is an odd integer. Equation (19) also implies that $(-e+\eta)$ is not in $(N \cup C)$, so

$$(-e+\eta)^{m''} - (-e+\eta)^{n''} \in N, \tag{22}$$

where m'', n'' are of *opposite* parity. Combining (19) and (22), we see that

$$(-e)^{m''} - (-e)^{n''} \in N, \tag{23}$$

and hence $2e \in N$, since m'' and n'' are of *opposite* parity. Therefore, $(2e)^{\gamma} = 0$, $\gamma \in \mathbb{Z}^+$, and thus $2^{\gamma}e = 0$, which implies that

$$2^{\gamma} e\eta \in C; \quad \gamma \in \mathbb{Z}^+. \tag{24}$$

Now, combining (21) and (24), keeping in mind that $(2^{\gamma}, m' - n') = 1$, we see that $e\eta \in C$, and hence, by (19), $\eta = e\eta \in C$, which contradicts (19). This contradiction proves the lemma. □

As usual, $[x,y] = xy - yx$ denotes the commutator of x and y.

We are now in a position to prove the following theorem.

Theorem 6. *Suppose R is a generalized periodic ring, and suppose that there exists an element c in C, with $c \neq 0$, such that*

$$c[x,y]=0 \quad \textit{implies } [x,y]=0, \ \forall x,y \in R. \tag{25}$$

Then R is commutative.

Proof. We distinguish two cases.
Case 1 ($c \in N$). In this case, $c^k = 0$ for some positive integer k, and hence

$$c^k[x,y]=0, \quad \forall x,y \in R. \tag{26}$$

Combining (25) and (26), we see that

$$\begin{aligned} c^k[x,y]=0 &\Longrightarrow c[c^{k-1}x,y]=0 \Longrightarrow [c^{k-1}x,y]=0 \Longrightarrow c^{k-1}[x,y]=0 \\ &\Longrightarrow \cdots \Longrightarrow c[x,y]=0 \Longrightarrow [x,y]=0. \end{aligned} \tag{27}$$

Thus, $c^k[x,y]=0$ implies $[x,y]=0$, and hence R is commutative.
Case 2 ($c \notin N$). In view of Theorem 3, we may assume that R is periodic. This implies, in particular, that c^m is idempotent for some positive integer m. Furthermore, $c^m \neq 0$ (since $c \notin N$ in our present case). The net result is (since $c \in C$ also)

$$c^m = e \quad \text{is a nonzero central idempotent in } R. \tag{28}$$

Let $a \in N$. By Lemma 5 and equation (28), we have $ea \in C$, and hence $[ea,x]=0$ for all $x \in R$, which implies

$$[c^m a, x] = c^m[a,x] = 0, \quad \forall x \in R. \tag{29}$$

The argument used in Case 1 of Theorem 6 shows that

$$c^m[a,x]=0 \quad \text{implies } [a,x]=0, \tag{30}$$

and hence (see (29))

$$[a,x]=0 \quad \forall x \in R, \ \forall a \in N. \tag{31}$$

Thus, R is a periodic ring with the property that $N \subseteq C$. By a well-known theorem of Herstein [4], it follows that R is commutative, and the theorem is proved. □

Corollary 7. *Suppose that R is a generalized periodic ring with identity 1. Then, R is commutative.*

Corollary 7 follows at once by taking $c = 1$ in Theorem 6.

Since a generalized Boolean ring is also a generalized periodic ring, therefore we have the following corollary.

Corollary 8. *A generalized Boolean ring with identity 1 is necessarily commutative.*

Another corollary is the following result, proved by the authors in [1].

Corollary 9. *Suppose that R is a generalized periodic ring containing a central element which is not a zero divisor. Then R is commutative.*

This follows at once, since the hypotheses of this corollary imply the hypotheses of Theorem 6.

Theorem 10. *Suppose R is a generalized periodic ring. Suppose, further, that there exists a nonzero central element c such that*

$$ca = 0 \quad \textit{implies } a = 0, \ \forall a \in N. \tag{32}$$

Then R is commutative.

Proof. In [1], the authors proved the following result:

$$\text{If } R \text{ is a generalized periodic ring, then the nilpotents } N \text{ form an ideal and } R/N \text{ is commutative.} \tag{33}$$

Let $x, y \in R$. By (33), for all $\bar{x}, \bar{y}$ in R/N, $\bar{x}\bar{y} = \bar{y}\bar{x}$, and hence $[x,y] \in N$. Taking $a = [x,y] \in N$ in (32), we see that (32) yields

$$c[x,y] = 0 \quad \text{implies } [x,y] = 0, \ \forall x, y \in R. \tag{34}$$

The theorem now follows at once from Theorem 6. □

Theorem 11. *A generalized Boolean ring R with central idempotents is necessarily nil $(R = N)$ or commutative $(R = C)$.*

Proof. Since R is also a generalized periodic ring, therefore by Theorem 3, R is commutative or periodic. If R is commutative, there is nothing to prove. So we may assume that R is periodic. We now distinguish two cases.

Case 1 ($C \subseteq N$). Recall that, by hypothesis, the set E of idempotents is central, and hence $E \subseteq C \subseteq N$ (in the present case). Thus, $E \subseteq N$, and hence $E = \{0\}$. Therefore,

$$\text{zero is the only idempotent of } R. \tag{35}$$

Let $x \in R$. Since R is periodic, therefore x^k is idempotent for some positive integer k, and hence by (35), $x^k = 0$, which proves that R is nil.

Case 2 ($C \nsubseteq N$). Then, for some $c \in R$, we have

$$c \in C, \quad c \notin N. \tag{36}$$

Again, since R is periodic, c^m is idempotent for some positive integer m. Moreover, $c^m \neq 0$ (since $c \notin N$). The net result is (see (36))

$$e = c^m \text{ is a nonzero central idempotent of } R. \tag{37}$$

Now, suppose $a \in N$. Since $0 \neq e \in C$ and $a \in N$, therefore $e + a \notin N$. Suppose, for the moment, that $a \notin C$. Then $e + a \notin C$ (since $e \in C$), and hence $e + a \notin (N \cup C)$. Therefore, by Definition 2,

$$(e+a) - (e+a)^n \in (N \cap C), \quad \text{for some even integer } n \geq 2. \tag{38}$$

Since R is also a generalized periodic ring, therefore by Lemma 5 (see (37))

$$ea^i \in C, \quad \forall i \in \{1,\dots,n-1\},\ (0 \neq e = e^2,\ e \in C,\ a \in N). \tag{39}$$

Combining (38) and (39), we see that

$$a - a^n \in C, \quad \forall a \in N \backslash C. \tag{40}$$

Since (40) is trivially satisfied for $a \in (N \cap C)$, therefore

$$a - a^n \in C, \quad \forall a \in N,\ n \geq 2. \tag{41}$$

We claim that

$$N \subseteq C. \tag{42}$$

The proof is by contradiction. Suppose (42) is false. Then, for some $a \in R$, we have

$$a \in N, \quad a \notin C. \tag{43}$$

Since $a \in N$, there exists a positive integer σ_0 such that

$$a^\sigma \in C, \quad \forall \sigma \geq \sigma_0,\ \sigma_0 \textit{ minimal}. \tag{44}$$

Moreover, since $a \notin C$ (see (43)), therefore $\sigma_0 > 1$. Now, applying (41) to the nilpotent element $a^{\sigma_0 - 1}$, we see that

$$a^{\sigma_0 - 1} - (a^{\sigma_0 - 1})^n \in C, \quad \text{for some } n = n(a^{\sigma_0 - 1}) \geq 2. \tag{45}$$

Furthermore, since $(\sigma_0 - 1)n \geq (\sigma_0 - 1)2 \geq \sigma_0$ (since $\sigma_0 \geq 2$), (44) implies that

$$(a^{\sigma_0 - 1})^n = a^{(\sigma_0 - 1)n} \in C. \tag{46}$$

Combining (45) and (46), we conclude that $a^{\sigma_0 - 1} \in C$, which contradicts the minimality of σ_0 in (44). This contradiction proves (42). Since R is a periodic ring satisfying (42), therefore, by a well-known theorem of Herstein [4], R is commutative. This completes the proof. □

Corollary 12. *A generalized Boolean ring with central idempotents and commuting nilpotents is commutative.*

This corollary recovers a result proved by the authors in [1].

Corollary 13. *If R is a generalized Boolean ring, and if R is 2-torsion-free, then R is nil or commutative.*

Proof. We claim that all idempotents of R are central. Suppose not, and suppose e is a noncentral idempotent in R. Then $-e \notin (N \cup C)$, and hence (see Definition 2)

$$(-e) - (-e)^n \in C, \quad n \textit{ even}. \tag{47}$$

Thus, $2e \in C$, and hence $[2e, x] = 0$ for all x in R. Since R is 2-torsion-free, $2[e,x] = 0$ implies $[e,x] = 0$, and thus $e \in C$, a contradiction. This contradiction proves that all idempotents of R are central, and hence R is nil or commutative, by Theorem 11. □

Theorem 14. *Let R be a generalized Boolean ring in which every finite subring is either commutative or nil. Then R is either commutative or nil.*

Proof. By contradiction. Thus, suppose R is a generalized Boolean ring such that every finite subring of R is either commutative or nil. Suppose, further, that R is not commutative and not nil either. By Theorem 11, there must exist a *noncentral* idempotent element e in R, and hence $e \notin (C \cup N)$. Thus (see Definition 2), since $-e \notin (C \cup N)$,

$$(-e) - (-e)^n \in (N \cap C), \quad n \text{ even}. \tag{48}$$

This implies that $2e \in (N \cap C)$, and hence $(2e)^k = 2^k e = 0$, for some $k \in \mathbb{Z}^+$. Since $e \notin C$, we must have the following:

$$\text{Either } ex - exe \neq 0 \text{ for some } x \in R, \text{ or } x'e - ex'e \neq 0 \text{ for some } x' \in R. \tag{49}$$

Suppose $u = ex - exe \neq 0$. Then,

$$eu = u \neq 0 = ue = u^2, \quad (u = ex - exe \neq 0). \tag{50}$$

Moreover,

$$2u = [2e, ex] = 0 \quad (\text{since } 2e \in C). \tag{51}$$

Furthermore, the subring generated by e and u is

$$\langle e, u \rangle = \{re + su \mid r,\ s \in \mathbb{Z}\}. \tag{52}$$

Since $2^k e = 0$ and $2u = 0$, the subring $\langle e, u \rangle$ is *finite*. Indeed,

$$\langle e, u \rangle = \{re + su \mid 1 \leq r \leq 2^k,\ 1 \leq s \leq 2\}. \tag{53}$$

On the other hand, if $x'e - ex'e \neq 0$ for some $x' \in R$ (the only other possibility), then the subring, $\langle e, v \rangle$, generated by e and $v = x'e - ex'e$ is (as is readily verified)

$$\langle e, v \rangle = \{re + sv \mid 1 \leq r \leq 2^k,\ 1 \leq s \leq 2\}. \tag{54}$$

Again, $\langle e, v \rangle$ is a *finite* subring of R. Hence, in either case, we found a *finite* subring of R, which is neither commutative (since $e \notin C$), nor nil (since $e \notin N$), contradicting our hypothesis. This contradiction proves the theorem. □

Remark 15. A careful examination of the proof of Theorem 14 shows that we only need to assume that "every subring S, with $|S| = 2^m$ for some positive integer m, is commutative or nil" in order for the ground generalized Boolean ring R to be commutative or nil. Indeed, $|\langle e, u \rangle| = 2^k \cdot 2 = 2^{k+1}$, since the representation of any x in this subring in the form $x = re + su$; $r, s \in \mathbb{Z}$, is *unique*. For, suppose $x = re + su$ and $x = r'e + s'u$. Then, $(r - r')e = (s' - s)u$. Recall that $2u = 0$, and $ue = 0$. Thus, if $s' - s$ is even, then $(r - r')e = 0$, and hence $re = r'e$, $su = s'u$. On the other hand, if $s' - s$ is odd, then $(r - r')e = u$, and hence $(r - r')ee = ue = 0$. Again, we obtain $re = r'e$, $su = s'u$.

We conclude with the following examples.

Example 16. Let

$$R=\left\{\begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a,b,c \in \mathrm{GF}(4)\right\}. \tag{55}$$

It is readily verified that the idempotents of R are central and

$$x-x^7=0, \quad \forall x \in R\backslash(N\cup C), \tag{56}$$

but R is neither nil nor commutative. Hence, Theorem 11 is not true if we drop the hypothesis that "n is *even*" in the definition of a generalized Boolean ring.

Example 17. Let

$$R=\left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a,b,c \in \mathrm{GF}(3)\right\}. \tag{57}$$

This example shows that we cannot drop the hypothesis that "N is commutative" in Corollary 12. (Note that R is not commutative.)

Example 18. Let

$$R=\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : 0,1 \in \mathrm{GF}(2)\right\}. \tag{58}$$

This example shows that we cannot drop the hypothesis that "the idempotents are central" in Corollary 12. (Note that R is not commutative.) This example also shows that we cannot drop the hypothesis that "R is 2-torsion-free" in Corollary 13. Note that, in this ring R, $x-x^2=0$ for all $x\in R\backslash(N\cup C)$. Even more is true. This ring R also shows that we cannot drop the hypothesis that "$1\in R$" in Corollary 7, nor the hypothesis that "$1\in R$" in Corollary 8.

Returning to the ring R in Example 16, we see that this ring further shows that we cannot drop the hypothesis that "m and n are of *opposite* parity" in the definition of a generalized periodic ring in connection with Corollary 7, or the hypothesis that "n is even" in the definition of a generalized Boolean ring as far as Corollary 8 is concerned. (Recall that $x-x^7=0$ for all $x\in R\backslash(N\cup C)$.)

Example 19. Let S be any *noncommutative* ring such that $S^3=(0)$. (For example, we may take S to be the ring of all 3×3 strictly upper triangular matrices over a field F.) Let $R=\mathrm{GF}(4)\oplus S$. It is readily verified that $x^3=x^6$ for all x in R, and hence R is indeed a generalized periodic ring. Moreover, the only idempotents of R are $(0,0)$ and $(1,0)$, and thus the idempotents of R are certainly central. Had R been a generalized Boolean ring, then, by Theorem 11, R would have to be either nil or commutative, which is certainly false here (recall that S is *not* commutative). This example shows that the set of generalized periodic rings is a wider class than that of generalized Boolean rings, and thus Theorem 11 does not hold for generalized periodic rings.

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