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PROPERTIES OF SOLUTIONS OF OPTIMIZATION PROBLEMS FOR SET FUNCTIONS

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ABSTRACT. A definition of a special class of optimization problems with set functions is given. The existence of optimal solutions and first-order optimality conditions are proved. This case of optimal problems can be transformed to standard mixed problems of mathematical programming in Euclidean space. It makes possible the applications of various algorithms for these optimization problems for finding conditional extrema of set functions.

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1. Introduction. We present selected methods of solving the optimization problems for set functions. These functions are the maps defined on the family of measurable subsets of a given space. The optimization problems for the set functions rely on finding its conditional extrema. Decision variables are the measurable sets, that is, the elements of M^k for some fixed $k \geq 1$, where M is a given σ-algebra. These problems are specific; they appear rather rarely in applications in comparison with "ordinary" problems of mathematical programming (with the functions defined on subsets of the Euclidean space). The optimization problems for set functions appear in some problems of mathematical statistics (estimation theory, robust analysis, testing hypothesis, for example, the problem of choosing the best critical set can be formulated as the problem of finding conditional extremum of special set functions, see, for example, [1, 5, 7]). Another class of problems with set functions can appear during the considerations of optimal traffic flows assignment in a given transportation network (cf. [3]).

This paper concentrates on computational aspects of solving optimization problems for set functions. The wide class of these problems is equivalent to the well-known problems of mathematical programming in Euclidean space (e.g., when the set functions have the form $F(S) = u(\int_S v)$, where u, v are given functions). This fact allowed to obtain existence theorem and first-order optimality conditions (Theorems 2.1 and 2.2). For these problems it is not necessary to construct new methods of finding solutions, because the known methods can be easily adapted and implemented.

The general case is much more complicated, because the domain of set functions usually does not have a linear, closed or convex structure. In this situation, the number of solving procedures (such as gradient algorithms) are less useful, but one can adapt and apply the evolution algorithms. Finally, we give some remarks about the possibility of implementing this kind of methods.

2. Optimization problems for set functions. Formulation of the problem and existence of solution. Let (X, M, μ) be a measurable space, that is, X is a nonempty set,

M is a σ-algebra of subsets of X, and $\mu : M \to \mathbb{R}$ is a bounded measure. Without loss of generality, we can assume that μ is nonnegative. A map $F : M^k \to R$ will be called a set function.

The main problem relies on determining the conditional extremum of a given set function.

Without loss of generality, we may assume that the problem is to determine the minimal value of a given set function $F_0 : M^k \to R$

$$F_0(S) \longrightarrow \min, \tag{2.1}$$

under some additional conditions (constraints)

(1) the conditions defined by the set functions $F_i : M^k \to R$, where $i = 1,\dots,s$

$$F_i(S) \le 0 \quad (i = 1,\dots,s); \tag{2.2}$$

(2) the conditions concerning directly the measurable sets formulated in terms of the characteristic functions of the sets from M

$$S \in M^k, \qquad \chi_S(x) \in V, \quad \text{for } \mu\text{-a.a. } x \in X, \tag{2.3}$$

where V is a given subset of $\{0,1\}^k$.

The problem (2.1), (2.2), and (2.3) represents a class of optimization problems. Only some of them can be effectively solved. In this paper, we consider the problems in which F_i $(i = 1,\dots,s)$ have the form

$$F_i(S) = u_i\left(\int_{S_1} v_1\,d\mu,\dots,\int_{S_k} v_k\,d\mu\right), \tag{2.4}$$

where $v_j : X \to R^q$, $v_i \in L^1(X,M,\mu)$ $(j = 1,\dots,k,\ q \in N)$.

Theorem 2.1. *The consistent problem (2.1), (2.1), and (2.3) (there exists at least one feasible solution), in which*

(1) *the functions F_i $(i = 0,1,\dots,s)$ have the form (2.4);*

(2) *u_0 is lower-semicontinuous, u_i $(i = 1,\dots,k)$ are continuous;*

has an optimal solution.

Proof. By denoting $a_j = \int_{S_j} v_j\,d\mu$ for $j = 1,\dots,k$,

$$\begin{gathered} W = \left\{\left(\int_{S_1} v_1\,d\mu,\dots,\int_{S_k} v_k\,d\mu\right) : S \in M^k,\ \chi_S(x) \in V \text{ for } \mu\text{-a.a. } x \in X\right\}, \\ T = \{x \in R^k : u_i(x) \le 0,\ \text{for } i = 1,\dots,s\}, \end{gathered} \tag{2.5}$$

the problem (2.1), (2.2), and (2.3) can be rewritten as

$$u_0(a_1,\dots,a_k) \longrightarrow \min, \tag{2.6}$$

subject to

$$(a_1,\dots,a_k) \in T \cap W. \tag{2.7}$$

By the generalized Lapunov convexity theorem if a measure μ is nonatomic, then the set of its values $\{\mu(S) : S \in M^k\}$ is convex (cf. [4]; some generalizations are proved in [3]). The set $T \cap W$ is nonempty and compact. The Weierstrass theorem gives the existence of the optimal solution $a^* = (a_1^*, \dots, a_k^*)$ of (2.6), (2.7). It follows from definition of W, in (2.5), that there exists an element $S^* = (S_1^*, \dots, S_k^*) \in M^k$, such that

$$a^* = \left(\int_{S_1^*} v_1 \, d\mu, \dots, \int_{S_k^*} v_k \, d\mu \right). \tag{2.8}$$

This element is obviously the optimal solution of the problem (2.1), (2.2), and (2.3). Of course, if u_0 is upper-semicontinuous, then the problem (2.1), (2.2), and (2.3) with maximization criterion in (2.1) has optimal solution. This completes the proof. □

For the wide class of problems (2.1), (2.2), and (2.3), where the set functions have the form (2.4), it is easy to obtain (from general theorems, see, for example, [2, 3]) the necessary conditions for optimality.

Theorem 2.2. (1) *If $S^* = (S_1^*, \dots, S_k^*)$ is the optimal solution of (2.1), (2.2), and (2.3), where F_i have the form (2.4), the measure μ is nonatomic, and the u_i $(i = 0, \dots, s)$ are differentiable, then there exist constants $\lambda_0^*, \dots, \lambda_s^* \geq 0$ such that, for $j = 1, \dots, k$ and a feasible solution S,*

$$\int_X \mathcal{F}(x) \cdot (\chi_S(x) - \chi_{S^*}(x)) \, d\mu(x) \geq 0, \tag{2.9}$$

where for $x \in X$,

$$\mathcal{F}(x) = \sum_{i=0}^{s} \lambda_i^* u_i' \left(\int_{S_1^*} v_1 \, d\mu, \dots, \int_{S_k^*} v_k \, d\mu \right) (v_1(x), \dots, v_k(x))'. \tag{2.10}$$

(2) *If conditions (2.2), (2.3) do not appear explicitly, then for $j = 1, \dots, k$,*

$$\begin{gathered} x \in S_j^* \Longrightarrow \frac{\partial u_0}{\partial x_j} \left(\int_{S_1^*} v_1 \, d\mu, \dots, \int_{S_k^*} v_k \, d\mu \right) v_j(x) \leq 0; \\ x \notin S_j^* \Longrightarrow \frac{\partial u_0}{\partial x_j} \left(\int_{S_1^*} v_1 \, d\mu, \dots, \int_{S_k^*} v_k \, d\mu \right) v_j(x) \geq 0; \\ u\left(t_{S_1^* - a_\mu} v_1 \, d\mu + \int_{S_1 \cap a_\mu} v_1 \, d\mu, \dots, \int_{S_k^* - a_\mu} v_k \, d\mu + \int_{S_k \cap a_\mu} v_k \, d\mu \right) \\ - u\left(\int_{S_1^*} v_1 \, d\mu, \dots, \int_{S_k^*} v_k \, d\mu \right) \geq 0. \end{gathered} \tag{2.11}$$

Example 2.3. If $u : R^q \to R$ is differentiable, then the problem

$$F_0(S) \longrightarrow \max(\min), \tag{2.12}$$

subject to

$$S \in M; \tag{2.13}$$

where

$$F_0 : M \longrightarrow R, \qquad F_0(S) = u\left(\int_S v_1 \, d\mu, \dots, \int_S v_q \, d\mu \right), \tag{2.14}$$

has an optimal solution (see Theorem 2.1). If S^* is optimal, then for any $S \in M$

$$x \in S^* \Longrightarrow f_{S^*,S\cap a_\mu}(x) \le 0; \qquad x \notin S^* \Longrightarrow f_{S^*,S\cap a_\mu}(x) \ge 0; \qquad \phi_{S^*}(S\cap a_\mu) \ge 0, \tag{2.15}$$

where

$$\begin{aligned} f_{S^*,S\cap a_\mu} &= \sum_{i=1}^{q} \frac{\partial u}{\partial x_i}\left(\int_{S^*-a_\mu} v_1\,d\mu + \int_{S\cap a_\mu} v_1\,d\mu, \dots, \int_{S^*-a_\mu} v_q\,d\mu + \int_{S\cap a_\mu} v_q\,d\mu\right) \cdot v_i|_{X-\mathcal{A}_\mu}, \\ \phi_{S^*}(S\cap a_\mu) &= u\left(\int_{S^*-a_\mu} v_1\,d\mu + \int_{S\cap a_\mu} v_1\,d\mu, \dots, \int_{S^*-a_\mu} v_q\,d\mu + \int_{S\cap a_\mu} v_q\,d\mu\right) \\ &\quad - u\left(\int_{S^*-a_\mu} v_1\,d\mu + \int_{S^*\cap a_\mu} v_1\,d\mu, \dots, \int_{S^*-a_\mu} v_q\,d\mu + \int_{S^*\cap a_\mu} v_q\,d\mu\right). \end{aligned} \tag{2.16}$$

We consider the special case where $X = [0,1]$, $M = \beta([0,1])$, μ is the sum of the Lebesgue measure on $[0,1]$, and the probability measure concentrated on $\{1\}$; the set function is given by the formula

$$F : M \longrightarrow R, \qquad F(S) = \int_S v_1(x)\,d\mu(x) + \left(\int_S v_2(x)\,d\mu(x)\right)^2, \tag{2.17}$$

with $v_1(x) = x$, $v_2(x) = 1-2x$ for $x \in [0,1]$.

We will solve the problems

$$F(S) \longrightarrow \max, (\min) \tag{2.18}$$

subject to $S \in M$. Both of these problems have, of course, optimal solutions. Let S^* denote the optimal solution of the problem of minimizing F. Theorem 2.2 gives

$$\begin{aligned} v_1(x) + 2\left(\int_{S^*} v_2\,d\mu\right) v_2(x) &\le 0 \quad \text{if } x \in S^* - \{1\}, \\ v_1(x) + 2\left(\int_{S^*} v_2\,d\mu\right) v_2(x) &\ge 0 \quad \text{if } x \in [0,1] - (S^* \cup \{1\}), \end{aligned} \tag{2.19}$$

and for any S

$$\begin{aligned} &\int_{S\cap\{1\}} v_1\,d\mu + \left(\int_{S^*-\{1\}} v_2\,d\mu + \int_{S\cap\{1\}} v_2\,d\mu\right)^2 \\ &\qquad \ge \int_{S^*\cap\{1\}} v_1\,d\mu + \left(\int_{S^*-\{1\}} v_2\,d\mu + \int_{S^*\cap\{1\}} v_2\,d\mu\right)^2. \end{aligned} \tag{2.20}$$

First we check if $x = 1$ belongs to S^*. If $\{1\} \subset S^*$, then by (2.20), it follows that for every S satisfying $S \cap \{1\} = \varnothing$ we have

$$0 + \left(\int_{S^*-\{1\}} v_2\,d\mu\right)^2 \ge 1 + \left(\int_{S^*-\{1\}} v_2\,d\mu - 1\right)^2, \tag{2.21}$$

hence

$$\int_{S^*-\{1\}} v_2\,d\mu \ge 1. \tag{2.22}$$

The last inequality cannot hold because the maximum value of $\int_A v_2\,d\mu$ (where $A = [0,1/2]$) is equal to $1/4$. This contradiction shows that $1 \notin S^*$.

Equation (2.19) now becomes ($d\xi$ denotes the Lebesgue measure on $[0,1]$)

$$\begin{aligned} &x+2(1-2x)\int_{S^*} v_2(\xi)\,d\xi \le 0 \quad \text{for } x \in S^*, \\ &x+2(1-2x)\int_{S^*} v_2(\xi)\,d\xi \ge 0 \quad \text{for } x \in [0,1]-(S^*\cup\{1\}). \end{aligned} \tag{2.23}$$

It is sufficient to search the solution S^* in the family of subintervals of $[0,1]$. Putting $S^* = [a,b]$, where $0 \le a \le b \le 1$, into (2.23) we obtain

$$x \in [a,b] \Longrightarrow x \le \frac{-2\Lambda}{1-4\Lambda}, \qquad x \in [0,1[-[a,b] \Longrightarrow x \ge \frac{-2\Lambda}{1-4\Lambda}, \tag{2.24}$$

where

$$\Lambda = \int_a^b (1-2\xi)\,d\xi = (b-a)-(b^2-a^2). \tag{2.25}$$

This clearly implies that $a = 0$. Moreover, two last implications are true for the set $S^* = [0,b]$, if b satisfies the equation

$$b = -2\frac{b-b^2}{1-4(b-b^2)}. \tag{2.26}$$

It has only one real solution $b = 0$. We conclude that F takes minimum value (equal 0) for $S^* = \{0\}$ and clearly for all subsets $[0,1]$ with null Lebesgue measure.

We find the maximum of F. Denote the set for which F takes a maximum value by S^{**}. The conditions are similar to (2.19) and (2.20) (with opposite signs of inequalities). It is easy to show that $1 \in S^{**}$.

From the formulas

$$\begin{aligned} x \in S^{**}-\{1\} &\Longrightarrow v_1(x)+2v_2(x)\int_{S^{**}} v_2\,d\mu \ge 0, \\ x \in [0,1]-S^{**} &\Longrightarrow v_1(x)+2v_2(x)\int_{S^{**}} v_2\,d\mu \le 0, \end{aligned} \tag{2.27}$$

substituting $S^{**} = [a,b] \subset [0,1]$ we obtain immediately that $a < 1$, $b = 1$ or $a = b = 1$. In the first case, a must satisfy the equation

$$\frac{2\Lambda}{4\Lambda-1} = a. \tag{2.28}$$

It has the unique solution $\bar{a}$ belonging to $[0,1]$

$$\bar{a} = 0{,}416,\ldots. \tag{2.29}$$

In the case when $a = b = 1$, the formula (2.28) is inconsistent. Finally,

$$S^{**} = [\bar{a},1], \tag{2.30}$$

and the maximum value of F is $F(S^{**}) = 2{,}958,\ldots.$

3. Selected aspects of solving optimization problems for set functions. We start this section by reformulating the problem (2.1), (2.2), and (2.3), or equivalently (2.6), (2.7).

Let $\mathcal{A}$ denote a family of atoms of measure μ, μ_{at}-section of μ into σ-algebra generated by $\mathcal{A}$ and $\mu_{\text{na}} = \mu - \mu_{\text{at}}$.

Defining, for $j = 1,\dots,k$,

$$t_j = \int_{S_j} v_j \, d\mu_{\text{na}}, \qquad \bar{t}_j = \int_{S_j} v_j \, d\mu_{\text{at}} = \sum_{a \in \mathcal{A}_\mu} v_j(a)\mu(a)x_{j,a}, \tag{3.1}$$

where for $a \in \mathcal{A}_\mu$

$$\begin{gathered} x_{j,a} = \begin{cases} 1 & \text{if } \mu(a - S_j) = 0, \\ 0 & \text{if } \mu(a - S_j) > 0, \end{cases} \\ W = \left\{ \left(\int_{S_1} v_1 \, d\mu_{na}, \dots, \int_{S_k} v_k \, d\mu_{na} \right) : S \in M^k,\ \chi_S(x) \in v(x) \text{ for } \mu\text{-a.a. } x \in X \right\}, \end{gathered} \tag{3.2}$$

we can transform the problem (2.1), (2.2), and (2.3), or equivalently (2.6), (2.7), to the following form

$$u_0(t_1 + \bar{t}_1, \dots, t_k + \bar{t}_k) \longrightarrow \min, \tag{3.3}$$

subject to

$$\begin{gathered} u_i(t_1 + \bar{t}_1, \dots, t_k + \bar{t}_k) \le 0, \quad i = 1,\dots,s,\ (t_1,\dots,t_k) \in W; \\ \bar{t}_j = \sum_{a \in \mathcal{A}_\mu} v_j(a)\mu(a)x_{j,a}, \quad x_{j,a} \in \{0,1\},\ j = 1,\dots,k,\ a \in \mathcal{A}_\mu. \end{gathered} \tag{3.4}$$

Moreover, denoting $\phi : \prod^k [0,1]^{\mathcal{A}_\mu} \to R$, we have

$$\phi\Big((x_{1,a})_{a \in \mathcal{A}_\mu}, \dots, (x_{1,a})_{a \in \mathcal{A}_\mu} \Big) = \left(\sum_{a \in \mathcal{A}_\mu} v(a)\mu(a)x_{1,a}, \dots, \sum_{a \in \mathcal{A}_\mu} v(a)\mu(a)x_{k,a} \right), \tag{3.5}$$

the problem (3.3), (3.4) may be written as

$$u_0(t_1 + \bar{t}_1, \dots, t_k + \bar{t}_k) \longrightarrow \min, \tag{3.6}$$

subject to

$$u_i(t_1 + \bar{t}_1, \dots, t_k + \bar{t}_k) \le 0, \quad i = 1,\dots,s,\ (t_1 + \bar{t}_1, \dots, t_k + \bar{t}_k) \in W + \phi(\{0,1\}^{\mathcal{A}_\mu}); \tag{3.7}$$

this problem has a countably set of decision variables, namely, k continuous variables t_j and $k|\mathcal{A}_\mu|$ binary variables $x_{j,a}$ (measure μ is bounded; this implies that its family of atoms is countable). Equations (3.3) and (3.4) are mixed problems of mathematical programming. The binary variables are therefore connected with the family of μ's atoms.

The relaxation is the problem in which the binary variables may take values from the interval $[0,1]$. This corresponds to replacing the last group of constraints in (3.3) and (3.4) by

$$x_{j,a} \in [0,1], \quad j = 1,\dots,k,\ a \in \mathcal{A}_\mu, \tag{3.8}$$

or equivalently—the last condition (3.7) is changed to

$$(t_1+\bar{t}_1,\dots,t_k+\bar{t}_k)\in W+\phi([0,1]^{\mathcal{A}_\mu}). \tag{3.9}$$

Example 3.1. Consider the problem solved in Example 2.3. The equivalent problem of mathematical programming has the following form

$$t_1+t_3+(t_2-t_3)^2 \longrightarrow \max(\min), \tag{3.10}$$

subject to

$$(t_1,t_3)\in W,\qquad t_3\in\{0,1\}, \tag{3.11}$$

where

$$W=\left\{\left(\int_S x\,dx,\int_S(1-2x)\,dx\right): S\in\beta([0,1])\right\}. \tag{3.12}$$

In this case it is relatively easy to determine the set W. It is sufficient to consider in (3.12) only the family of intervals $S=[a,b]$, where $0\le a\le b\le 1$. Hence

$$W=\left\{\left(\frac{1}{2}(b^2-a^2),b-a-(b^2-a^2)\right): 0\le a\le b\le 1\right\}. \tag{3.13}$$

Solving the system

$$t_1=\frac{1}{2}(b^2-a^2),\qquad t_2=b-a-(b^2-a^2), \tag{3.14}$$

we obtain

$$a=\frac{1}{2}\left(\frac{2t_1}{2t_1+t_2}-(2t_1+t_2)\right),\qquad b=\frac{1}{2}\left(\frac{2t_1}{2t_1+t_2}+(2t_1+t_2)\right). \tag{3.15}$$

The constraints $0\le a\le b\le 1$ lead to the set of conditions defining W

$$2t_1+t_2\ge 0,\quad 2(2t_1+t_2)\ge(2t_1+t_2)^2+2t_1,\quad 2t_1\ge(2t_1+t_2)^2,\quad t_1\ge 0. \tag{3.16}$$

The problem (2.1), (2.2), and (2.3) with the set functions having the form (2.4) can be solved by the methods for mixed integer mathematical programming. In most cases, it is very difficult to determine the shape of the set W (or $\tilde{W}$). Because this set is convex (see Lapunov convexity theorem), it can be approximated by convex hull generated by finite number of its elements: if $S^{(1)},\dots,S^{(m)}$ is a sequence of sets satisfying (2.3), then W can be (for large m) replaced, with sufficient precision, by (the symbol $\operatorname{conv}(B)$ denotes the convex hull of the set B, that is, the set of all finite convex combinations of the elements of B)

$$\operatorname{conv}\left(t(S^{(1)}),\dots,t(S^{(m)})\right), \tag{3.17}$$

where

$$t(S^{(p)})=\left(\int_{S_1^{(p)}}v_1\,d\mu,\dots,\int_{S_k^{(p)}}v_k\,d\mu\right). \tag{3.18}$$

Of course, the above construction is based on the specific form of the problem (2.1), (2.2), and (2.3) with (2.4). Below we present an appropriate construction in the general case (without the assumption (2.4)). Unfortunately, it seems that the procedure is less effective and it is not rather helpful in finding optimal solutions. We introduce the measurable space $(\tilde{X},\tilde{M},\tilde{\mu})$, where

$$\tilde{X} = X\times[0,1], \qquad \tilde{M} = \Big\{(A-a_\mu)\times\{0\}\cup\bigcup_{a\in\mathcal{A}_\mu} a\times B_a : A\in M,\ B_a\in\beta([0,1])\Big\}, \tag{3.19}$$

$\tilde{\mu}:\tilde{M}\to R$ is defined by

$$\tilde{\mu}\Big(A\times\{0\}\cup\bigcup_{a\in\mathcal{A}_\mu} a\times B_a\Big) = \mu(A)+\sum_{a\in\mathcal{A}_\mu}\mu(a)\lambda(B_a). \tag{3.20}$$

In (3.19) and (3.20), $\beta([0,1])$ denotes the family of Borel subsets of $[0,1]$, λ is the Lebesgue measure on $[0,1]$, $\mathcal{A}_\mu$ is the family of atoms of μ and a_μ is the sum of all sets from $\mathcal{A}_\mu$.

Note that condition (3.9) can be written as

$$(t_1+\bar{t}_1,\dots,t_k+\bar{t}_k)\in\tilde{W}, \tag{3.21}$$

where

$$\begin{aligned}\tilde{W} = \Big\{\Big(\int_{S_1}\tilde{v}_1\,d\tilde{\mu},\dots,\int_{S_k}\tilde{v}_k\,d\tilde{\mu}\Big) : \tilde{S} = (S_1,\dots,S_k)\in\tilde{M}^k,\\ \chi_{(S_1,\dots,S_k)}(x,t)\in\tilde{v}(x) \text{ for } \tilde{\mu}\text{- a.a. } (x,t)\in\tilde{X}\Big\},\end{aligned} \tag{3.22}$$

where pr_1 is a projection on X in $X\times[0,1]$ and $\tilde{v}_j = v_j\circ\mathrm{pr}_1$ for $j=1,\dots,k$.

Note that it is possible to construct a relaxation for any problem (2.1), (2.2), and (2.3). It suffices to replace the functions F_i $(i=0,\dots,s)$ in (2.1), (2.2), and (2.3) by other functions $\tilde{F}_i$ defined on $\tilde{M}$ with the following conditions.

(1) If $\lambda(B_{a_i})=1$, for $i=1,\dots,n$, then

$$\tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n} a_i\times B_{a_i}\Big) = F\Big(A\cup\bigcup_{i=1}^{n} a_i\Big). \tag{3.23}$$

(2) For any $\{a_1,\dots,a_n\}\subset\mathcal{A}_\mu$

$$\begin{aligned}\tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n} a_i\times B_{a_i}\Big) &= \tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n-1} a_i\times B_{a_i}\Big)(1-\lambda(B_{a_n}))\\ &\quad+\lambda(B_{a_n})\tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n-1} a_i\times B_{a_i}\cup a_n\times[0,1]\Big).\end{aligned} \tag{3.24}$$

(3) For any permutation π of the set $\{1,\dots,n\}$

$$\tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n} a_i\times B_{a_i}\Big) = \tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n} a_{\pi(i)}\times B_{a_{\pi(i)}}\Big). \tag{3.25}$$

(4) For any element of $\tilde{M}^k$

$$\tilde{F}\Big(A\times\{0\}\cup\bigcup_{a\in\mathcal{A}_\mu} a\times B_a\Big) = \liminf_{n\to+\infty}\tilde{F}\Big(A\times\{0\}\cup\bigcup_{i=1}^{n} a_i\times B_{a_i}\Big). \tag{3.26}$$

Other kinds of problems may appear in the case when u_i in (2.4) are not sufficiently smooth (continuous, convex, differentiability, etc.). Fortunately, even when the difficulties with applying the methods using differentiability appear, one can use evolution algorithms. These procedures can be applied even if the set functions do not have the form (2.4). Suitable transformation of the initial problem with a proper structure of data lead to a solution with arbitrary precision.

Let $\Delta = \{\Delta_t : t = 1,\dots,T\}$ be a measurable partition of X (the family Δ satisfy the conditions $\Delta_t \in M$ for $t = 1,\dots,T$, $X = \bigcup_{t=1}^{T} \Delta_t$ and $\mu(\Delta_{t_1} \cap \Delta_{t_2}) = 0$ for $t_1 \neq t_2$). The approximation depend on replacing the condition $S \in M^k$ in (2.3) by $S \in \sigma(\Delta)^k$, where $\sigma(\Delta)$ denotes the σ-algebra generated by the collection Δ. The feasible solution S of (2.1), (2.2), and (2.3) may be encoded as the map $\phi : \{1,\dots,T\} \to V$ is defined as

$$\phi(t) = v \iff \forall_{j=1,\dots,k} (\Delta_t \subset S_j \iff v_j = 1). \tag{3.27}$$

To determine an approximation of the element $S^* \in \sigma(\Delta)^k$, in which (2.1) takes the minimum one can apply the genetic algorithm. The chromosomes can be identified with the maps ϕ. Set functions F_i correspond to the maps defined on chromosomes for $S \in \sigma(\Delta)^k$ and $i = 0,1,\dots,s$, we define $\hat{F}_i$ by

$$\hat{F}_i(\phi) = F_i(S) \iff (\phi, S) \quad \text{satisfy (3.27)}. \tag{3.28}$$

The problem (2.1), (2.2), and (2.3) should be replaced by

$$\hat{F}_0(\phi) + FK(\phi) \longrightarrow \min, \tag{3.29}$$

subject to $\phi : \{1,\dots,T\} \to V$, in which the function $FK(\cdot)$ protects against violating the constraints (2.2). This function can be given by the following formula

$$FK(\phi) = m \cdot \max_{i=1,\dots,k} (0, \hat{F}_i(\phi)), \tag{3.30}$$

where m denotes a great positive number. The family of functions ϕ admits such operations as selection, mutation (inversion) and crossing. It is easy to see that these operations lead to feasible solutions of the problem (3.29) (the full review of another operation that can be defined on the chromosomes can be found for example in [6]). The key-problem is the proper choice of the family $\sigma(\Delta)$. It follows from the necessary conditions for optimality, that it should be finer than the σ-algebra generated by the sets $\bigcap_{j=1}^{k} v_j^{-1}(B)$, where B is any Borel subset of R.

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