

EFFECTS OF POISEUILLE FLOW ON PERISTALTIC TRANSPORT

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ABSTRACT. The effects of Poiseuille flow on peristaltic transport in a cylindrical tube has been investigated. A perturbation solution is obtained, which satisfies the momentum equation for the case in which the amplitude ratio (wave amplitude/tube radius) is chosen as a small parameter. The results show that the fluid phase mean axial velocity increases with increasing the Poiseuille parameter k . The phenomena of reflux is discussed. Numerical results are reported for various values of k .

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1. Introduction. The term “peristalsis” is used for the mechanism by which a fluid can be transported through a distensible tube when contraction or expansion waves propagate progressively along its length. Peristaltic action is used by human organs (such as ureter, male reproductive system, gastro-intestinal tract, bile duct) to perform their functions effectively.

Latham [10] was probably the first to investigate the mechanism of peristalsis in relation to mechanical pumping. Since then, several investigators have contributed to the study of peristaltic action in both mechanical and physiological situations, such as Burns et al. [3], Hanin [8], Barton et al. [1], Fung et al. [7], Shapiro et al. [16], Yih et al. [19], Chow [4], Jaffrin et al. [9], and Mittra [13].

In particular, Burns and Parkes used perturbation techniques to study the peristaltic motion through a channel and a tube. Barton and Raylor studied the peristaltic motion in a circular tube by using long and short wave length approximations. The fluid mechanics of the ureter has been studied by Lykoudis et al. [11] and Boyarsky et al. [2]. The interaction of Poiseuille flow with the peristaltic motion has been studied by Mittra et al. [14].

Several theoretical and experimental attempts have been made to understand peristaltic action in different situations. A summary of most of the experimental and theoretical investigations reported with details of the geometry, fluid, Reynolds number, wave length parameter, wave amplitude parameter, and wave shape have been given by Srivastava et al. [17]. Srivastava et al. [18] studied the effects of Poiseuille flow on peristaltic transport of a particular suspension. Saxena et al. [15] studied the particular suspension flow induced by sinusoidal peristaltic waves. In another type of studies on peristaltic transport, El Misery et al. [5] studied the peristaltic motion of an incompressible generalized Newtonian fluid in a planar channel. Mekheimer et al. [12] studied the peristaltic motion of a particle fluid suspension in a planar channel. El Shehawey et al. [6] studied the couple-stress in peristaltic transport of fluids. In the case of $k = 0$, our result is in agreement with Yih et al. [19].

2. Formulation of the problem. We consider a two-dimensional circular cylindrical tube of radius R , with axisymmetric, moderate-amplitude traveling waves imposed on its wall.

The fluid is assumed to be Newtonian, viscous, homogeneous, and incompressible, occupying a semi-infinite region of the space. The equations governing two-dimensional motion of a viscous incompressible fluid are

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right), \quad (2.1a)$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right), \quad (2.1b)$$

and the equation of continuity is

$$\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r} = 0, \quad (2.2)$$

where z is the axial coordinate in the direction of wave propagation, r is the radial coordinate, ν is the kinematic viscosity, ρ is the density, P is the pressure, t is the time, and v_r and v_z are the velocity components in the r - and z -directions, respectively. Axisymmetric motion is assumed.

The velocity components can be written in terms of Stoke's stream function Ψ as

$$v_r = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad v_z = -\frac{1}{r} \frac{\partial \Psi}{\partial r}. \quad (2.3)$$

Using (2.3) to eliminate the pressure P between (2.1a) and (2.1b) yields the equation for the stream function

$$\frac{\partial}{\partial t} (\bar{\nabla}^2 \Psi) + \frac{1}{r} \frac{\partial \Psi}{\partial z} \left[\bar{\nabla}^2 \left(\frac{\partial \Psi}{\partial r} \right) - \frac{2}{r} \bar{\nabla}^2 \Psi + \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right] - \frac{1}{r} \frac{\partial \Psi}{\partial r} \bar{\nabla}^2 \left(\frac{\partial \Psi}{\partial z} \right) = \nu \bar{\nabla}^2 \bar{\nabla}^2 \Psi, \quad (2.4)$$

where $\bar{\nabla}^2$ is a special operator defined as

$$\bar{\nabla}^2 \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}. \quad (2.5)$$

At the boundaries, the fluid is subjected to the motion of the wall in the form (see Figure 2.1),

$$\eta = a \cos \frac{2\pi}{\lambda} (z - ct), \quad (2.6)$$

where a is the amplitude, η is the radial displacement from the mean position of the wall, λ is the wavelength, and c is the wave speed. The boundary conditions that must be satisfied by the fluid on the walls are the no-slip and impermeability conditions. The velocity components of the fluid particles on the wall are thus

$$v_z = 0, \quad v_r = \frac{\partial \eta}{\partial t}, \quad (2.7)$$

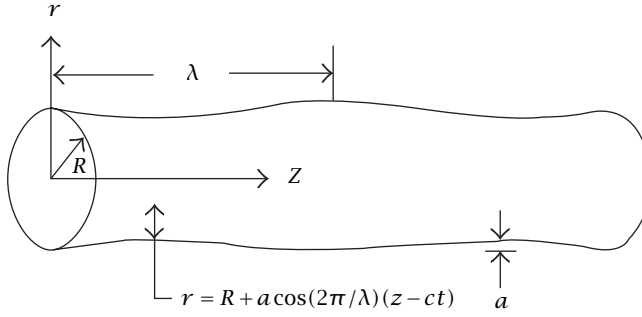


FIGURE 2.1 Geometry of the problem.

and in terms of the stream function Ψ ,

$$\frac{\partial \Psi}{\partial r} = 0, \quad \frac{\partial \Psi}{\partial z} = \frac{2\pi a c r}{\lambda} \sin \frac{2\pi}{\lambda} (z - ct). \quad (2.8)$$

We introduce the nondimensional variables and parameters as follows:

$$\begin{aligned} r^* &= \frac{r}{R}, & z^* &= \frac{z}{R}, & v_r^* &= \frac{v_r}{c}, & v_z^* &= \frac{v_z}{c}, \\ \eta^* &= \frac{\eta}{R}, & \Psi^* &= \frac{\Psi}{R^2 c}, & t^* &= \frac{ct}{R}, & P^* &= \frac{p}{\rho c^2}, \end{aligned} \quad (2.9a)$$

and we define the amplitude ratio ϵ , the wave number α , and the Reynolds number Re as

$$\epsilon = \frac{a}{R}, \quad \alpha = \frac{2\pi R}{\lambda}, \quad Re = \frac{cR}{\nu}. \quad (2.9b)$$

The nondimensional forms of (2.4), (2.6), and (2.8), after dropping the star are

$$\frac{\partial}{\partial t} (\nabla^2 \Psi) + \frac{1}{r} \frac{\partial \Psi}{\partial z} \left[\nabla^2 \frac{\partial \Psi}{\partial r} - \frac{2}{r} \nabla^2 \Psi + \frac{1}{r^2} \frac{\partial \Psi}{\partial r} \right] - \frac{1}{r} \frac{\partial \Psi}{\partial r} \nabla^2 \frac{\partial \Psi}{\partial z} = \frac{1}{Re} \nabla^2 \nabla^2 \Psi, \quad (2.10)$$

$$\eta = \epsilon \cos \alpha(z - t), \quad (2.11)$$

$$\frac{\partial \Psi}{\partial r} = 0, \quad \frac{\partial \Psi}{\partial z} = \alpha \epsilon r \sin \alpha(z - t), \quad r = 1 + \eta. \quad (2.12)$$

3. Method of solution. We expand Ψ and $\partial P / \partial z$ in a power series of the small parameter ϵ ,

$$\Psi = \Psi_0 + \epsilon \Psi_1 + \epsilon^2 \Psi_2 + \dots, \quad (3.1)$$

$$\frac{\partial P}{\partial z} = \left(\frac{\partial P}{\partial z} \right)_0 + \epsilon \left(\frac{\partial P}{\partial z} \right)_1 + \epsilon^2 \left(\frac{\partial P}{\partial z} \right)_2 + \dots. \quad (3.2)$$

Substituting (3.1) into (2.10), collecting terms of equal powers of ϵ , and equating the coefficients of like powers on both sides of the equation, we obtain

$$\frac{1}{\text{Re}} \bar{\nabla}^2 \bar{\nabla}^2 \Psi_0 = \frac{\partial}{\partial t} \bar{\nabla}^2 \Psi_0 + \frac{1}{r} \frac{\partial \Psi_0}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_0}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_0 + \frac{1}{r^2} \frac{\partial \Psi_0}{\partial r} \right] - \frac{1}{r} \frac{\partial \Psi_0}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_0}{\partial z}, \quad (3.3)$$

$$\begin{aligned} \frac{1}{\text{Re}} \bar{\nabla}^2 \bar{\nabla}^2 \Psi_1 &= \frac{\partial}{\partial t} \nabla^2 \Psi_1 + \frac{1}{r} \frac{\partial \Psi_1}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_0}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_0 + \frac{1}{r^2} \frac{\partial \Psi_0}{\partial r} \right] \\ &+ \frac{1}{r} \frac{\partial \Psi_0}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_1}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_1 + \frac{1}{r^2} \frac{\partial \Psi_1}{\partial r} \right] \\ &- \frac{1}{r} \frac{\partial \Psi_1}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_0}{\partial z} - \frac{1}{r} \frac{\partial \Psi_0}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_1}{\partial z}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \frac{1}{\text{Re}} \bar{\nabla}^2 \bar{\nabla}^2 \Psi_2 &= \frac{\partial}{\partial t} \bar{\nabla}^2 \Psi_2 + \frac{1}{r} \frac{\partial \Psi_2}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_0}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_0 + \frac{1}{r^2} \frac{\partial \Psi_0}{\partial r} \right] \\ &+ \frac{1}{r} \frac{\partial \Psi_1}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_1}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_1 + \frac{1}{r^2} \frac{\partial \Psi_1}{\partial r} \right] \\ &+ \frac{1}{r} \frac{\partial \Psi_0}{\partial z} \left[\bar{\nabla}^2 \frac{\partial \Psi_2}{\partial r} - \frac{2}{r} \bar{\nabla}^2 \Psi_2 + \frac{1}{r^2} \frac{\partial \Psi_2}{\partial r} \right] \\ &- \frac{1}{r} \frac{\partial \Psi_2}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_0}{\partial z} - \frac{1}{r} \frac{\partial \Psi_1}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_1}{\partial z} - \frac{1}{r} \frac{\partial \Psi_0}{\partial r} \bar{\nabla}^2 \frac{\partial \Psi_2}{\partial z}. \end{aligned} \quad (3.5)$$

We expand the boundary conditions (2.12) in power series of η ,

$$\begin{aligned} \Psi_r(1) + \eta \Psi_{rr}(1) + \frac{\eta^2}{2} \Psi_{rrr}(1) + \dots &= 0, \\ \Psi_z(1) + \eta \Psi_{rz}(1) + \frac{\eta^2}{2} \Psi_{zrr}(1) + \dots &= \alpha \epsilon \sin \alpha(z-t). \end{aligned} \quad (3.6)$$

Substituting (2.11) and (3.1) into (3.6) and equating coefficients of like powers of ϵ on both sides of the equation, we obtain

$$\frac{\partial \Psi_0}{\partial r}(1) = 0, \quad (3.7)$$

$$\frac{\partial \Psi_1}{\partial r}(1) + \frac{\partial^2 \Psi_0}{\partial r^2}(1) \cos \alpha(z-t) = 0, \quad (3.8)$$

$$\frac{\partial \Psi_2}{\partial r}(1) + \frac{\partial^2 \Psi_1}{\partial r^2}(1) \cos \alpha(z-t) + \frac{1}{2} \frac{\partial^3 \Psi_0}{\partial r^3}(1) \cos^2 \alpha(z-t) = 0, \quad (3.9)$$

$$\frac{\partial \Psi_0}{\partial z}(1) = 0, \quad (3.10)$$

$$\frac{\partial \Psi_1}{\partial z}(1) + \frac{\partial^2 \Psi_0}{\partial r \partial z}(1) \cos \alpha(z-t) = \alpha \sin \alpha(z-t), \quad (3.11)$$

$$\frac{\partial \Psi_2}{\partial z}(1) + \frac{\partial^2 \Psi_1}{\partial r \partial z}(1) \cos \alpha(z-t) + \frac{1}{2} \frac{\partial^3 \Psi_0}{\partial r^2 \partial z}(1) \cos^2 \alpha(z-t) = 0, \quad (3.12)$$

Equations (3.3), (3.7), and (3.10), together with a condition of uniform pressure gradient $(\partial P / \partial z)_0 = \text{constant}$, are satisfied which yields the classical Poiseuille flow for a fluid,

$$\Psi_{0r} = k(2r - 2r^3), \quad (3.13)$$

where

$$k = \frac{\text{Re}}{8} \left(\frac{\partial P}{\partial z} \right)_0 \quad (3.14)$$

is the Poiseuille flow parameter for a fluid. Equation (3.4) with their corresponding boundary conditions (3.8) and (3.11) shows that a solution can be chosen in the form

$$\Psi_1 = \phi_1(r)e^{i\alpha(z-t)} + \phi_1^*(r)e^{-i\alpha(z-t)}, \quad (3.15)$$

where the asterisk denotes a complex conjugate.

Substituting (3.15) into (3.4), (3.8), and (3.11), we obtain the Sommerfeld-Orr equation with the corresponding boundary conditions as follows:

$$\left[\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - \alpha^2 + i\alpha \text{Re} [1 + 2k(1-r^2)] \right] \left(\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - \alpha^2 \right) \phi_1 = 0, \quad (3.16)$$

$$\phi_1'(1) = 2k, \quad \phi_1(1) = -\frac{1}{2}, \quad (3.17)$$

where the prime denote the differentiation with respect to r .

Equation (3.5), with its corresponding boundary conditions (3.9) and (3.12), shows that a solution can be chosen in the form

$$\Psi_2 = \phi_{20}(r) + \phi_{22}(r)e^{2i\alpha(z-t)} + \phi_{22}^*(r)e^{-2i\alpha(z-t)}. \quad (3.18)$$

Substituting (3.18) into (3.5), (3.9), and (3.12), we obtain the two equations for ϕ_{20} and ϕ_{22} with the corresponding boundary conditions as follows:

$$\begin{aligned} \bar{\nabla}^2 \bar{\nabla}^2 \phi_{20} = \frac{i\alpha \text{Re}}{r} \left[-\frac{3}{r} \phi_1 \phi_1^{*''} + \phi_1 \phi_1^{*''''} + \frac{3}{r^2} \phi_1 \phi_1^{*'} \right. \\ \left. + \frac{3}{r} \phi_1' \phi_1^* - \phi_1^* \phi_1''' - \frac{3}{r^2} \phi_1' \phi_1^* + \phi_1' \phi_1^{*''} - \phi_1^{*'} \phi_1'' \right], \end{aligned} \quad (3.19)$$

$$\begin{aligned} \tilde{\nabla}^2 \tilde{\nabla}^2 \phi_{22} = -2i\alpha \text{Re} [1 + 2k(1-r^2)] \tilde{\nabla}^2 \phi_{22} \\ + \frac{i\alpha \text{Re}}{r} \left[\phi_1 \phi_1''' - \phi_1' \phi_1'' - \frac{3}{r} \phi_1 \phi_1'' + \frac{1}{r} (\phi_1')^2 + \frac{3}{r^2} \phi_1 \phi_1' + \frac{2}{r} \alpha^2 (\phi_1)^2 \right] \end{aligned} \quad (3.20)$$

with

$$\bar{\nabla}^2 = \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr}, \quad \tilde{\nabla}^2 = \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - 4\alpha^2, \quad (3.21)$$

$$\phi_{20}'(1) + \frac{1}{2} [\phi_1''(1) + \phi_1^{*''}(1)] = 3k, \quad (3.22)$$

$$\phi_{22}'(1) + \frac{1}{2} \phi_1''(1) = \frac{3}{2}k, \quad \phi_{22}(1) + \frac{1}{4} \phi_1'(1) = 0. \quad (3.23)$$

Thus, we obtain a set of differential equations together with the corresponding boundary conditions which is sufficient to determine the solution of the problem up to the second order in ϵ .

Now, our main purpose is to find out the solutions of differential equations for ϕ_1 , although (3.16) for ϕ_1 is the fourth-order ordinary differential equation with variable

coefficients. However, we can restrict our investigation to the case in which an initially stagnant fluid is subject to imposed pressure gradient. Thus, in this case $(\partial P/\partial z)_0 = \text{constant}$, which means that the constant k does not vanish and we would be able to obtain a simple analytical solution of this case.

4. Analysis. We consider the case in which the pressure gradient $(\partial P/\partial z)_0 = \text{constant}$. In this case, there will be flow if the wall motion stops. Hence, $k = \text{constant}$, and (3.16) becomes

$$\left(\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - \gamma^2\right)\left(\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} - \alpha^2\right)\phi_1 = 0, \tag{4.1}$$

for which $\gamma^2 = \alpha^2 - i\alpha \text{Re}(1 + 2k(1 - r^2))$ and the boundary conditions (3.17) become

$$\phi_1'(1) = 2k, \quad \phi_1(1) = -\frac{1}{2}. \tag{4.2}$$

Transformation of (4.1) by the Hankel transform $H(s)$, where

$$H(s) = \int_{-\infty}^{\infty} r f(r) J_n(sr) dr, \tag{4.3}$$

the boundary conditions (4.2) and the condition that the velocity must remain finite at $r = 0$, besides using the inversion formula of the Hankel transform, lead to the solution

$$\phi_1(r) = c_1 r I_0(\beta r) + c_3 r I_1(\alpha r), \tag{4.4}$$

where $\beta^2 = \alpha^2 - i\alpha \text{Re}(1 + 2k)$,

$$c_1 = \frac{-\alpha I_0(\alpha) - 4k I_1(\alpha)}{2(\alpha I_0(\alpha) I_1(\beta) - \beta I_0(\beta) I_1(\alpha))}, \quad c_3 = \frac{\beta I_0(\beta) + 4k I_1(\beta)}{2(\alpha I_0(\alpha) I_1(\beta) - \beta I_0(\beta) I_1(\alpha))}, \tag{4.5}$$

where I_0 and I_1 are the modified Bessel functions of the first kind.

By substituting (4.4) and its conjugate into (3.19) and (3.22), we obtain

$$\begin{aligned} \bar{\nabla}^2 \bar{\nabla}^2 \phi_{20} = & -\alpha^2 \text{Re}^2(1 + 2k(1 - r^2)) \\ & \times [2c_1 c_1^* [r \beta^* I_1(\beta r) I_0(\beta^* r) + r \beta I_0(\beta r) I_1(\beta^* r) - 2I_1(\beta r) I_1(\beta^* r)] \\ & + c_1 c_3^* [r \beta I_0(\beta r) I_1(\alpha r) + r \alpha I_1(\beta r) I_0(\alpha r) - 2I_1(\beta r) I_1(\alpha r)] \\ & + c_1^* c_3 [r \beta^* I_1(\alpha r) I_0(\beta^* R) + r \alpha I_0(\alpha r) I_1(\beta^* r) - 2I_1(\beta^* r) I_1(\alpha r)], \end{aligned} \tag{4.6}$$

$$\begin{aligned} \phi_{20}'(1) = & -\frac{1}{2} [c_1 \beta^2 I_1(\beta) + (c_3 + c_3^*) \alpha^2 I_1(\alpha) + c_1^* \beta^{*2} I_1(\beta^*) \\ & + c_1 \beta I_0(\beta) + c_1^* \beta^* I_0(\beta^*) + (c_3 + c_3^*) \alpha I_0(\alpha)] + 3k = D. \end{aligned} \tag{4.7}$$

The right-hand side of (4.6) is a complicated function of r . We evaluate the right-hand side numerically and represent the result approximately by a polynomial of the following form:

$$\bar{\nabla}^2 \bar{\nabla}^2 \phi_{20} = -\alpha^2 \text{Re}^2(1 + 2k) \sum_{i=1}^S B_i r^{2i} + 2k \alpha^2 \text{Re}^2 \sum_{i=1}^S B_i r^{2i+2}, \tag{4.8}$$

for different values of α and Re , different B_i must be used. Solving (4.8), we obtain

$$\phi'_{20} = L_1 r + L_2 r^3 - \alpha^2 \text{Re}^2 (1 + 2k) \sum_{i=1}^S \frac{B_i r^{2i+3}}{2i(2i+2)^2} + 2k\alpha^2 \text{Re}^2 \sum_{i=1}^S \frac{B_i r^{2i+5}}{(2i+2)(2i+4)^2}, \quad (4.9)$$

where L_1 and L_2 are constants.

Defining the following functions:

$$G(r) = \sum_{i=1}^S \frac{B_i r^{2i+3}}{2i(2i+2)^2}, \quad H(r) = \sum_{i=1}^S \frac{B_i r^{2i+5}}{(2i+2)(2i+4)^2}, \quad (4.10)$$

we can write the boundary condition (4.7) in the form

$$\phi'_{20}(1) = L_1 + L_2 - \alpha^2 \text{Re}^2 (1 + 2k)G(1) + 2k\alpha^2 \text{Re}^2 H(1). \quad (4.11)$$

By substituting (4.10) and (4.11) into (4.9), we get

$$\begin{aligned} \phi'_{20}(r) = & [\alpha^2 \text{Re}^2 (1 + 2k)G(1) + D - 2k\alpha^2 \text{Re}^2 H(1)]r + L_2(r^3 - r) \\ & - \alpha^2 \text{Re}^2 (1 + 2k)G(r) + 2k\alpha^2 \text{Re}^2 H(r). \end{aligned} \quad (4.12)$$

If each term of (3.1), (3.2), (3.15), (3.18), and (4.4) is time-averaged over one period, we obtain the mean pressure gradient

$$\begin{aligned} \left(\frac{\partial \bar{p}}{\partial z} \right) = & \frac{\epsilon^2}{r \text{Re}} \left[-\phi''_{20} + \frac{1}{r} \phi'_{20} - \frac{1}{r^2} \phi'_{20} \right] \\ & + \frac{i\alpha\epsilon^2}{r^2} \left[\phi_1 \phi_1^{*''} - \phi_1^* \phi_1'' - \frac{1}{r} (\phi_1 \phi_1^{*'} - \phi_1^* \phi_1') \right] + \frac{8k}{\text{Re}} + O(\epsilon^3). \end{aligned} \quad (4.13)$$

By substituting (4.12) into (4.13) yields

$$\begin{aligned} \left(\frac{\partial \bar{p}}{\partial z} \right) = & \left[-\alpha^2 \text{Re} \left[-\frac{G''(r)}{r} + \frac{1}{r^2} G'(r) - \frac{1}{r^3} G(r) \right] + 2k\alpha^2 \text{Re}^2 \left[-\frac{H''(r)}{r} + \frac{H'(r)}{r^2} - \frac{H(r)}{r^3} \right] \right. \\ & \left. - \alpha^2 \text{Re}(1 + 2k) \left[-2c_1 c_1^* I_1(\beta r) I_1(\beta^* r) - c_1 c_3^* I_1(\beta r) I_1(\alpha r) - c_1^* c_3 I_1(\alpha r) I_1(\beta^* r) \right] \right] \\ & + \frac{8k}{\text{Re}} - \frac{4L_2}{\text{Re}} = Z + \frac{4}{\text{Re}} [2k - L_2]. \end{aligned} \quad (4.14)$$

The time-averaged pressure gradient is not constant across the tube, but has a perturbation which varies with the radius. This perturbation function, which is designated by Z , consists of the terms in the brackets in (4.14).

The solution for the mean axial velocity (averaged over time) is

$$\begin{aligned} \bar{v}_z = & -2k(1 - r^2) - \frac{1}{r} \epsilon^2 \phi'_{20} \\ = & \epsilon^2 \left[-\alpha^2 \text{Re}^2 (1 + 2k) \left[G(1) - \frac{G(r)}{r} \right] - D \right. \\ & \left. + 2k\alpha^2 \text{Re}^2 \left[H(1) - \frac{H(r)}{r} \right] - \left(\frac{2k}{\epsilon^2} - L_2 \right) (1 - r^2) \right]. \end{aligned} \quad (4.15)$$

The mean-axial velocity may be written in the form

$$\begin{aligned} \bar{v}_z = \epsilon^2 \left[-\alpha^2 \text{Re}^2 (1+2k) \left[G(1) - \frac{G(r)}{r} \right] - D \right. \\ \left. + 2k\alpha^2 \text{Re}^2 \left[H(1) - \frac{H(r)}{r} \right] - \frac{\text{Re}}{4} \left(\frac{\partial P}{\partial z} \right) (1-r^2) \right]. \end{aligned} \quad (4.16)$$

From (4.15) a “critical reflux condition” can be defined as the condition $\bar{v}_z = 0$ at $r = 0$, that is, (at the centre of the tube) using (4.14), this condition is

$$\left(\frac{\partial P}{\partial z} \right)_{\text{cr}} = \frac{4}{\text{Re}} (2k - L_2), \quad (4.17)$$

where

$$L_2 = \alpha^2 \text{Re}^2 (1+2k)G(1) - 2k\alpha^2 \text{Re}^2 H(1) + D + \frac{2k}{\epsilon^2}. \quad (4.18)$$

The mean flow rate can be easily found by integrating (4.15) over the cross-section which yields

$$\begin{aligned} \bar{q} = -\pi\epsilon^2 \left[\frac{k}{\epsilon^2} + \alpha^2 \text{Re}^2 (1+2k)G(1) - 2k\alpha^2 \text{Re}^2 H(1) - \frac{L_2}{2} + D \right] \\ + 2\pi\epsilon^2 \alpha^2 \text{Re}^2 \left[(1+2k) \int_0^1 G(r) dr - 2k \int_0^1 H(r) dr \right]. \end{aligned} \quad (4.19)$$

5. Numerical results and discussion. In order to study the behavior of the solution, numerical calculations for several values of α , k , and Re were carried out. Equation (4.15) shows that the mean axial velocity is dominated by the D term and the parabolic term

$$\left(L_2 - \frac{2k}{\epsilon^2} \right) (1-r^2). \quad (5.1)$$

The constant D initially arose from the nonslip conditions of the axial velocity on the wall and is related to the mean velocity at the boundary of the tube (at $r = 1$) by $\bar{v}_z = \epsilon^2 D$. The parabolic term is due to the time-averaged pressure gradient set up by the peristaltic motion. In addition to the two terms mentioned above, the velocity has a perturbation term which is a function of r , that is,

$$\left(G(1) - \frac{G(r)}{r} \right). \quad (5.2)$$

The variation of D with α for various values of Re and k is depicted in Figures 5.1 and 5.2. The numerical results show that D increases with increasing k .

The effects of k on the mean-velocity distribution and reversal flow are displayed in Figures 5.3, 5.4, 5.5, and 5.6. The results reveal that the mean velocity distribution increases proportionally with the Poiseuille parameter k . We notice that as k increases the velocity increases forward for $\partial P/\partial z < 0$, while for $\partial P/\partial z > 0$ the velocity increases backward (increasing the reflux flow).

For $\partial P/\partial z < (\partial P/\partial z)_{\text{cr}}$, there is no reflux and if $\partial P/\partial z > (\partial P/\partial z)_{\text{cr}}$ there will be reflux and a backward flow in the neighbourhood of the centre line occurs. We call a flow reflux whenever there is a region in the tube where the mean axial-velocity is opposite to the direction of the peristaltic waves.

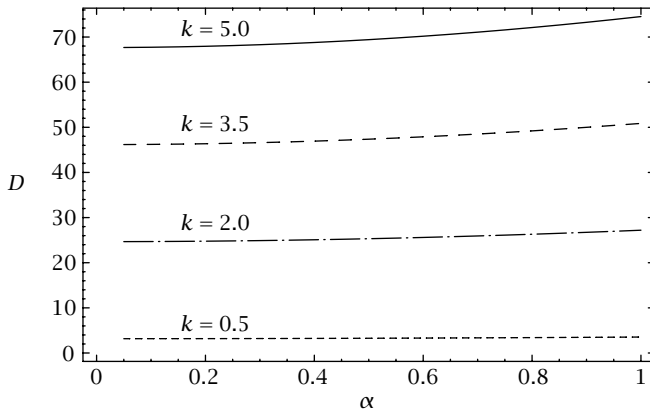


FIGURE 5.1 Variation of D with wave number α and Poiseuille parameter k at $\text{Re} = 1.0$.

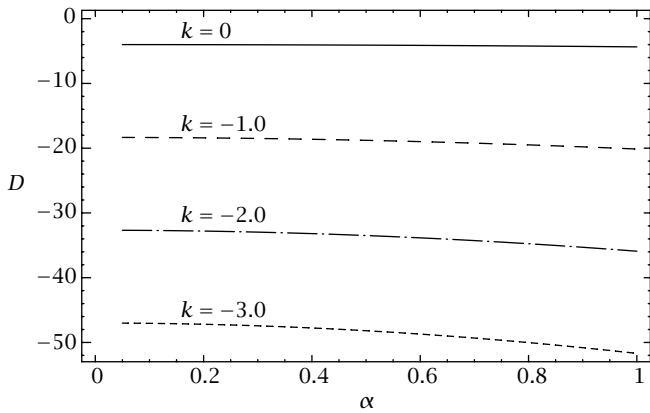


FIGURE 5.2 Variation of D with wave number α and Poiseuille parameter k at $\text{Re} = 1.0$.

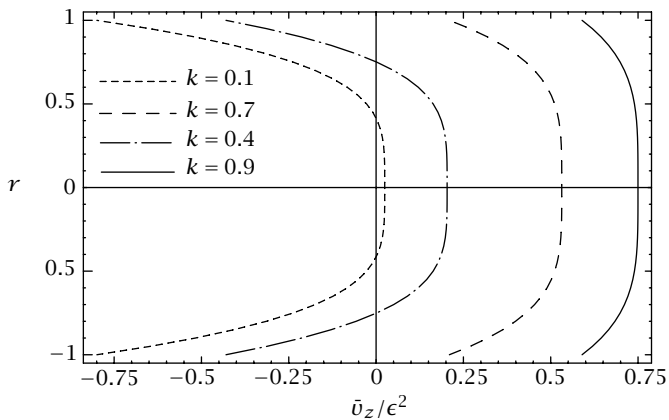


FIGURE 5.3 The effect of Poiseuille parameter k on the mean-velocity distribution and reversal flow at $\text{Re} = 5.0$ and $\epsilon = 0.15$.

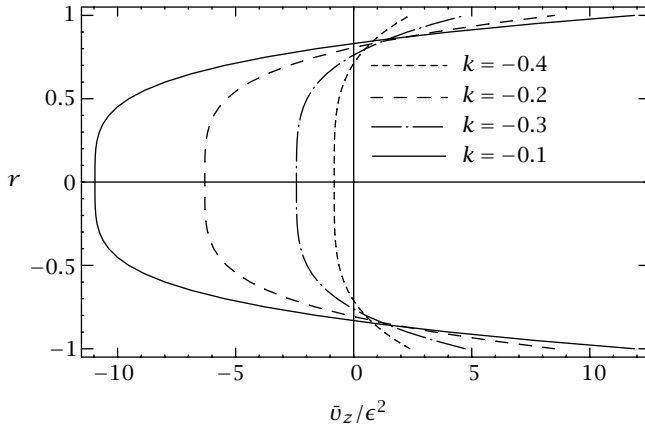
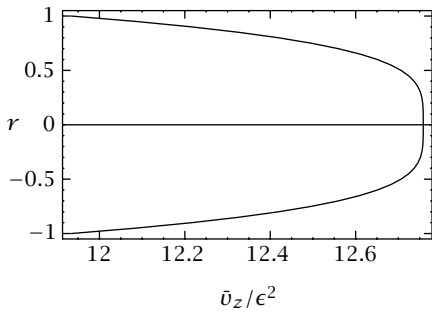
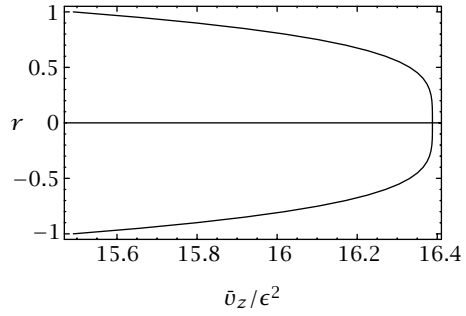


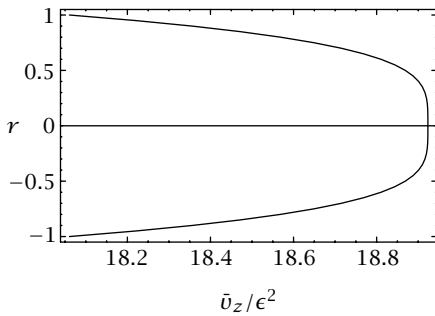
FIGURE 5.4 The effect of Poiseuille parameter k on the mean-velocity distribution and reversal flow at $Re = 5.0$ and $\epsilon = 0.15$.



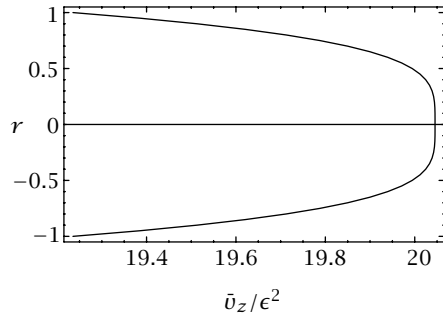
(a) $k = 0.2$.



(b) $k = 0.3$.



(c) $k = 0.6$.



(d) $k = 0.8$.

FIGURE 5.5 The effect of Poiseuille parameter k on the mean-velocity distribution and reversal flow at $Re = 5.0$, $\epsilon = 0.15$, and $(\partial P/\partial z) = -3.0$.

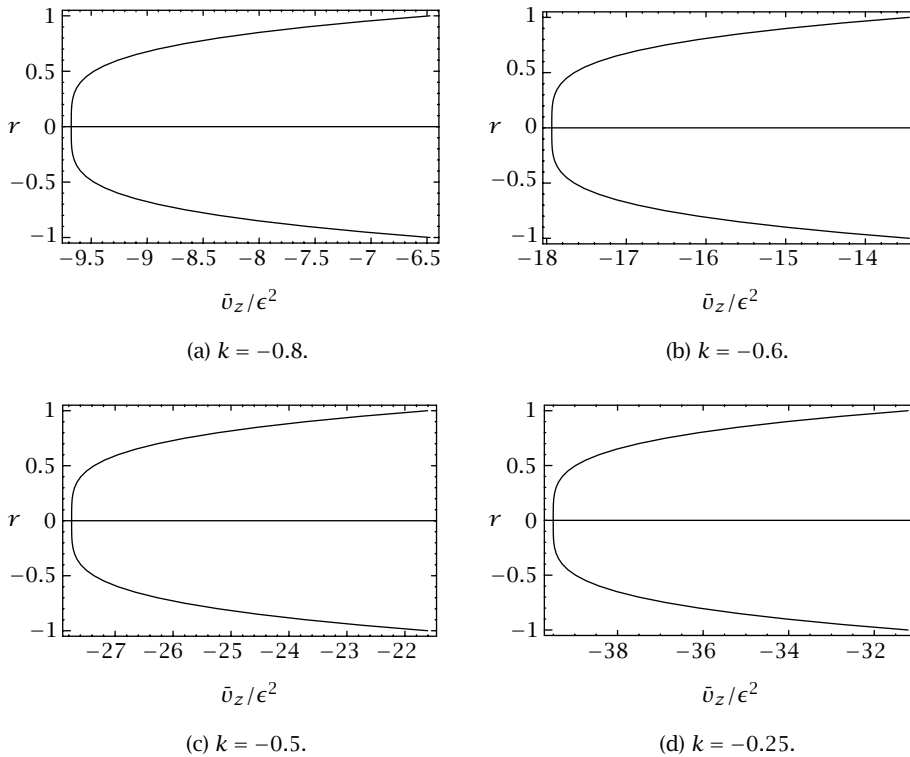


FIGURE 5.6 The effect of Poiseuille parameter k on the mean-velocity distribution and reversal flow at $Re = 5.0$, $\epsilon = 0.15$, and $(\partial P/\partial z) = 10.0$.

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