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COMPACTITY IN NARROW LIMIT TOWER SPACES

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Abstract. We introduce a limit tower structure on the space of all bounded Radon measures on a completely regular space and we extend the Prohorov's theorem of narrow compactness. In the particular case of Polish spaces, we give a sequential version of this extension.

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1. Introduction. Let T be a completely regular space, $\mathcal{B}$ the boreliens of T, and $\mathfrak{M}^b(T)$ the set of all bounded Radon measures on $(T,\mathcal{B})$ (i.e., the real bounded measures $\mu:\mathcal{B}\to\mathbb{R}$ such that $|\mu|(A)=\sup\{|\mu|(K):K$ is compact, $K\subseteq A\}$, for all $A\in\mathcal{B}$, where $|\mu|$ is the variation of μ). Denote by $\mathcal{C}^b(T)$ the space of all bounded continuous real functions on T and let $\|f\|=\sup\{|f(t)|:t\in T\}$, for every $f\in\mathcal{C}^b(T)$. We recall that a filter $\mathfrak{F}$ on $\mathfrak{M}^b(T)$ is narrowly convergent to μ if and only if $V_{\varepsilon,f}(\mu)\in\mathfrak{F}$, for all $f\in\mathcal{C}^b(T)$, $\varepsilon>0$, where $V_{\varepsilon,f}(\mu)=\{\nu:|\mu(f)-\nu(f)|<\varepsilon\}$.

We say that a set $H\subseteq\mathfrak{M}^b(T)$ is *relatively narrowly compact* if, for every filterbase $\mathfrak{B}\subseteq 2^H$ there exist a filter $\mathfrak{F}$ on $\mathfrak{M}^b(T)$ and $\mu\in\mathfrak{M}^b(T)$ such that $\mathfrak{F}$ converges to μ.

Prohorov's classical theorem states that *a bounded set $H\subseteq\mathfrak{M}^b(T)$ is relatively narrowly compact if the following condition is satisfied*:

$$\forall\varepsilon>0,\quad \exists K_\varepsilon \text{ compact}\subseteq T:|\mu|(T\setminus K_\varepsilon)<\varepsilon,\ \forall\mu\in H. \tag{1.1}$$

A set H as in (1.1) is called *tight*.

We remark that, if T is a Polish space (i.e., T is a separable, completely metrizable space) or T is a locally compact space, the converse is also true (relative narrow compactness implies tightness) (see [2, Section 5, Theorems 1 and 2]).

Limit tower spaces were first defined in 1997 by Kent and Brock [3] as an isomorphic gradated variant of convergence approach spaces of Löwen [8].

In this paper, we introduce on $\mathfrak{M}^b(T)$ a limit tower structure $\bar{p}=\{p_a:a\in[0,+\infty]\}$ (see [3]), where p_0 is the narrow convergence structure. Then, for every bounded set $H\subseteq\mathfrak{M}^b(T)$, there exists a number $t=t(H)\geq 0$ such that, for every filterbase $\mathfrak{B}\subseteq 2^H$ there exists a filter $\mathfrak{F}$ on $\mathfrak{M}^b(T)$, $\mathfrak{B}\subseteq\mathfrak{F}$, p_t-convergent in $\mathfrak{M}^b(T)$ (see Theorem 3.8); we say that H is *p_t-relatively compact*. The number $t(H)$ estimates the degree of tightness of H. If H is tight then $t(H)=0$, so we obtain Prohorov's theorem.

If T is a locally compact space we extend also the converse of Prohorov's theorem (see Theorem 3.12).

We give some examples in the particular case of $T=\mathbb{N}$ when $\mathfrak{M}^b(\mathbb{N})=\ell^1$.

In Section 4, we obtain a sequential version of Theorem 3.8 on the subset $\mathfrak{M}^1(T) \subseteq \mathfrak{M}^b(T)$ of all probabilities on the Polish space T. So, every sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M}^1(T)$ contains a subsequence p_{2t}-convergent in $\mathfrak{M}^1(T)$, where $t = t(\{\mu_n : n \in \mathbb{N}\})$ (see Theorem 4.9). In particular, we prove that the limit tower structure $\bar{p}$ on the set of probabilities is induced by a probabilistic metric on this space.

2. Limit tower structures. Let X be a set, $\mathbb{B}(X)$ the set of all filterbases on X, and 2^X the power set of X; for every $\mathfrak{f} \in \mathbb{B}(X)$, $\mathfrak{f}'$ is the filter generated by $\mathfrak{f}$. For $x \in X$, let $\dot{x}$ denote the fixed ultrafilter generated by $\{x\}$.

Definition 2.1 (see [3, Definition 1]). A *limit structure* on X is a function $q : \mathbb{B}(X) \to 2^X$ satisfying

$$x \in q(\dot{x}), \quad x \in X, \tag{2.1}$$

$$q(\mathfrak{f}) = q(\mathfrak{f}'), \quad \forall \mathfrak{f} \in \mathbb{B}(X), \tag{2.2}$$

$$q(\mathfrak{f}' \cap \mathfrak{G}') = q(\mathfrak{f}) \cap q(\mathfrak{G}), \quad \forall \mathfrak{f}, \mathfrak{G} \in \mathbb{B}(X). \tag{2.3}$$

A pair (X, q), where q is a limit structure on X is called a *limit space*.

Remark 2.2. The statement "$x \in q(\mathfrak{f})$" will be written $\mathfrak{f} \xrightarrow{q} x$ and we say that $\mathfrak{f}$ q-converges to x.

Remark 2.3. In [3], a *limit structure* is a function $q : \mathbb{F}(X) \to 2^X$, where $\mathbb{F}(X)$ denotes the set of all filters on X, satisfying

$$\begin{aligned} & x \in q(\dot{x}), \quad x \in X, \\ & \mathfrak{f} \subseteq \mathfrak{G} \Longrightarrow q(\mathfrak{f}) \subseteq q(\mathfrak{G}), \\ & x \in q(\mathfrak{f}) \Longrightarrow x \in q(\mathfrak{f} \cap \dot{x}), \\ & x \in q(\mathfrak{f}) \cap q(\mathfrak{G}) \Longrightarrow x \in q(\mathfrak{f} \cap \mathfrak{G}). \end{aligned} \tag{2.4}$$

If we extend such a function q to $\mathbb{B}(X)$ letting $q(\mathfrak{f}) = q(\mathfrak{f}')$, then (2.4) is equivalent to (2.1), (2.2), and (2.3).

Remark 2.4. If τ is a topology on X and we define $\mathfrak{f} \xrightarrow{q_\tau} x$ if and only if $\mathcal{V}_\tau(x) \subseteq \mathfrak{f}'$, then q_τ is a limit structure on X (here $\mathcal{V}_\tau(x)$ denotes the neighborhood filter of x in (X, τ)). More exactly we have the following proposition.

Proposition 2.5 (see [3, Proposition 2]). *Let q be a limit structure on X; the necessary and sufficient condition for a topology τ to exist on X, such that $q = q_\tau$, is that q fulfills the following condition:*

(F) *Let $\{\mathfrak{f}_j : j \in J\}$ be a family of filterbases on X and $\{x_j : j \in J\} \subset X$ be such that $\mathfrak{f}_j \xrightarrow{q} x_j$ for all $j \in J$.*

If $\Phi \in \mathbb{B}(J)$ is such that $\mathfrak{f} \xrightarrow{q} x$, where $\mathfrak{f} = \{\{x_j : j \in \phi\} : \phi \in \Phi\}$, then

$$\bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{f}'_j \xrightarrow{q} x. \tag{2.5}$$

Definition 2.6 (see [3, Definition 4]). A *limit tower* $\bar{p}$ on a set X is a family $\bar{p} = \{p_a : a \in [0,+\infty]\}$ of limit structures on X satisfying the following conditions:

$$p_a(\mathfrak{F}) \subseteq p_b(\mathfrak{F}), \quad \forall a \le b, \ \forall \mathfrak{F} \in \mathbb{B}(X), \tag{2.6}$$

$$p_\infty(\mathfrak{F}) = X, \quad \forall \mathfrak{F} \in \mathbb{B}(X), \tag{2.7}$$

$$p_a(\mathfrak{F}) = \cap_{b>a} p_b(\mathfrak{F}), \quad \forall a \in [0,+\infty), \ \forall \mathfrak{F} \in \mathbb{B}(X). \tag{2.8}$$

If $x \in p_a(\mathfrak{F})$, we say that $\mathfrak{F}$ is p_a-convergent to x and we denote this by $\mathfrak{F} \xrightarrow{p_a} x$. If $\bar{p}$ is a limit tower on X, $(X,\bar{p})$ is called a *limit tower space*.

The axiom (F) defined in Proposition 2.5 has a natural extension to a limit tower space $(X,\bar{p})$:

(F) Let $a,b \in [0,+\infty]$, $\{\mathfrak{F}_j : j \in J\} \subseteq \mathbb{B}(X)$, and $\{x_j : j \in J\} \subseteq X$ such that $\mathfrak{F}_j \xrightarrow{p_a} x_j$, for all $j \in J$. If $\Phi \in \mathbb{B}(J)$ is such that $\mathfrak{F} \xrightarrow{p_b} x$, where $\mathfrak{F} = \{\{x_j : j \in \phi\} : \phi \in \Phi\}$, then

$$\bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{F}'_j \xrightarrow{p_{a+b}} x. \tag{2.9}$$

Definition 2.7. A limit tower $\bar{p}$ on X which satisfies (F) is called a *topological limit tower*.

Remark 2.8. From [3, Theorems 9, 13 and Proposition 12(b)] we know that a topological limit tower is an isomorphic form of a Löwen's approach structure (see [8]).

3. Narrow limit tower on $\mathfrak{M}^b(T)$. In this section, we introduce a topological limit tower $\bar{p} = \{p_a : a \in [0,+\infty]\}$ on the space of bounded Radon measures on a completely regular space such that p_0-convergence is just the narrow convergence; then we extend the Prohorov's theorem of narrow compactness.

Let T be a completely regular space, let $\mathcal{B}$ be the σ-algebra of Borel subsets of T, and let $\mathfrak{M}^b(T)$ be the set of all bounded Radon measures on $(T,\mathcal{B})$. Denote by $C^b(T)$ the set of all bounded continuous real functions on T. For every $f \in C^b(T)$ and $\mu \in \mathfrak{M}^b(T)$, we denote $\mu(f) = \int_T f\,d\mu$.

Now, for every $a \in [0,+\infty]$, $\mu \in \mathfrak{M}^b(T)$, and $f \in C^b(T)$, we denote

$$V_{a,f}(\mu) = \Big\{\nu \in \mathfrak{M}^b(T) : \big|\mu(f) - \nu(f)\big| \le a\|f\|\Big\}. \tag{3.1}$$

Then, for every $a \in [0,+\infty)$, let $p_a : \mathbb{B}(\mathfrak{M}^b(T)) \to 2^{\mathfrak{M}^b(T)}$ defined by

$$p_a(\mathfrak{F}) = \Big\{\mu \in \mathfrak{M}^b(T) : \forall b > a, \forall \in C^b(T), V_{b,f}(\mu) \in \mathfrak{F}'\Big\}, \tag{3.2}$$

for all filterbases $\mathfrak{F}$ on $\mathfrak{M}^b(T)$; let p_∞ be the indiscrete convergence structure on $\mathfrak{M}^b(T)$ ($p_\infty(\mathfrak{F}) = \mathfrak{M}^b(T)$, for all $\mathfrak{F} \in \mathbb{B}(\mathfrak{M}^b(T))$).

We remark that $\mathfrak{F} \xrightarrow{p_a} \mu$ if and only if for all $b > a$, for all $f \in C^b(T), V_{b,f}(\mu) \in \mathfrak{F}'$.

Proposition 3.1. *The limit tower $\bar{p} = \{p_a : a \in [0,+\infty]\}$ is a topological limit tower on $\mathfrak{M}^b(T)$.*

Proof. For every $a \in [0,+\infty)$, $\mu \in \mathfrak{M}^b(T)$, and $f \in C^b(T)$, we have $\mu \in V_{a,f}(\mu)$, so that we have (2.1).

Equations (2.2), (2.3), (2.6), and (2.7) are consequences of the definition of $\bar{p}$. From (2.6), $p_a(\mathfrak{F}) \subseteq \bigcap_{b>a} p_b(\mathfrak{F})$, for all $\mathfrak{F} \in \mathbb{B}(\mathfrak{M}^b(T))$. If $\mathfrak{F} \xrightarrow{p_c} \mu$, for all $c > a$, then for all $b > a$, there exists c such that $a < c < b$ hence $V_{b,f}(\mu) \in \mathfrak{F}$, for all $f \in C^b(T)$. Therefore $\mathfrak{F} \xrightarrow{p_a} \mu$ and so we have (2.8).

(F) Let $a, b \geq 0, \{\mathfrak{F}_j : j \in J\} \subseteq \mathbb{B}(\mathfrak{M}^b(T))$, and $\{\mu_j : j \in J\} \subseteq \mathfrak{M}^b(T)$ such that (1) $\mathfrak{F}_j \xrightarrow{p_a} \mu_j$, for all $j \in J$. Let Φ be a filterbase on J such that (2) $\mathfrak{F} \xrightarrow{b} \mu$, where $\mathfrak{F} = \{\{\mu_j\}_{j\in\phi} : \phi \in \Phi\}$. Then for all $u > a+b$, there exist $d > a$, $e > b$ such that $u = d+e$. Then for all $f \in C^b(T)$, from (2), $V_{e,f}(\mu) \in \mathfrak{F}'$ hence, there exists $\phi \in \Phi$ such that $\{\mu_j\}_{j\in\phi} \subseteq V_{e,f}(\mu)$. Then (3) $|\mu_j(f) - \mu(f)| \leq e\|f\|$, for all $j \in \phi$.

From (1), for all $j \in J$, $V_{d,f}(\mu_j) \in \mathfrak{F}'_j$. But, from (3), $V_{d,f}(\mu_j) \subseteq V_{u,f}(\mu)$, so that $V_{u,f} \in \mathfrak{F}'_j$, for all $j \in \phi$. Therefore,

$$V_{u,f}(\mu) \in \bigcap_{j\in\phi} \mathfrak{F}'_j \subseteq \bigcup_{\phi\in\Phi} \bigcap_{j\in\phi} \mathfrak{F}'_j. \tag{3.3}$$

It follows that $\mu \in p_{a+b}(\bigcup_{\phi\in\Phi} \bigcap_{j\in\phi} \mathfrak{F}'_j)$, so that $\bar{p} = \{p_a : a \in [0,+\infty]\}$ is a topological limit tower on $\mathfrak{M}^b(T)$. □

Definition 3.2. We say that $\bar{p} = \{p_a : a \in [0,+\infty]\}$ is the *narrow limit tower* on $\mathfrak{M}^b(T)$.

Remark 3.3. Note that p_0 is the narrow convergence structure on $\mathfrak{M}^b(T)$. Indeed, $\mathfrak{F} \xrightarrow{p_0} \mu$ if and only if for all $\varepsilon > 0$, for all $f \in C^b(T), V_{\varepsilon,f} \in \mathfrak{F}'$. But the sets $V_{\varepsilon,f}(\mu) = \{\nu : |\mu(f) - \nu(f)| \leq \varepsilon\|f\|\}$ form a subbase for the neighbourhood system of μ in the narrow topology on $\mathfrak{M}^b(T)$; so that $\mathfrak{F}$ is narrowly convergent to μ.

Remark 3.4. If $\mathfrak{F} \xrightarrow{p_a} \mu$ then $\mathfrak{F} \xrightarrow{p_b} \mu$, for all $b \geq a$. Thus p_0 is the finest limit structure of $\bar{p}$.

Remark 3.5. We may interpret $\inf\{a : \mathfrak{F} \xrightarrow{p_a} \mu\}$ as the degree of narrow convergence of filterbase $\mathfrak{F}$ to μ.

Remark 3.6. For every net $(\mu_i)_{i\in I} \subseteq \mathfrak{M}^b(T)$ ($(I, \leq)$ is a directed set) let $\mathfrak{F} = \{\{x_j : j \geq i\} : i \in I\}$ be the filterbase generated by $(\mu_i)_{i\in I}$.

If $\bar{p} = \{p_a : a \in [0,+\infty]\}$ is the narrow limit tower on $\mathfrak{M}^b(T)$, then we say that $\mu_i \xrightarrow{a} \mu$ if $\mathfrak{F} \xrightarrow{p_a} \mu$. Therefore, $\mu_i \xrightarrow{a} \mu$ if and only if

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a \cdot \|f\|, \quad \forall f \in C^b(T). \tag{3.4}$$

Definition 3.7. We say that a subset $H \subseteq \mathfrak{M}^b(T)$ is *a-relatively compact* if for every filterbase $\mathfrak{B} \subseteq 2^H$, there exist a filter $\mathfrak{F}$ on $\mathfrak{M}^b(T)$ and $\mu \in \mathfrak{M}^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{F}$ and $\mathfrak{F} \xrightarrow{p_a} \mu$.

We remark that H is 0-relatively compact if and only if H is relatively narrowly compact.

A subset $H \subseteq \mathfrak{M}^b(T)$ is bounded if $\sup\{|\mu|(T) : \mu \in H\} < +\infty$, where $|\mu|$ is the variation of μ. The mapping $\mu \mapsto |\mu|(T) = \|\mu\|$ is a norm on $\mathfrak{M}^b(T)$.

Let $\mathcal{K}(T)$ be the family of all compact sets on T; for every bounded set $H \subseteq \mathfrak{M}^b(T)$

we denote

$$t(H) = \inf_{K \in \mathcal{K}(T)} \sup_{\mu \in H} |\mu|(T \setminus K). \tag{3.5}$$

We remark that $t(H) \in [0, +\infty)$ and $t(H) = 0$ if and only if H is tight. We say that $t(H)$ is the *degree of tightness* of H.

Now we give an extension of Prohorov's theorem.

Theorem 3.8. *Every bounded set $H \subseteq \mathfrak{M}^b(T)$ is $t(H)$-relatively compact.*

Proof. Let X be the Stone-Čech compactification of T and $i : T \to X$ be the canonical injection of T in X. We remark that $C^b(X) = C(X)$ (X is compact); so $(\mathfrak{M}^b(X), \|\cdot\|)$ is the topological dual of the Banach space $(C(X), \|\cdot\|)$ and the narrow topology on $\mathfrak{M}^b(X)$ is the weak*-topology, w^*, of this dual space.

For every $\mu \in \mathfrak{M}^b(T)$ we define $\nu = I(\mu) \in \mathfrak{M}^b(X)$, where $I(\nu)(F) = \mu(F \circ i)$, for every $F \in C(X)$; $\|\nu\| = |\nu|(X) = |\mu|(T) = \|\mu\|$ so that $I : \mathfrak{M}^b(T) \to \mathfrak{M}^b(X)$, $\mu \mapsto I(\mu)$, is an isometric embedding.

Let H be a bounded subset of $\mathfrak{M}^b(T)$; then $I(H)$ is a bounded subset of $\mathfrak{M}^b(X)$. Therefore $I(H)$ is w^*-relatively compact. For every filterbase $\mathfrak{B} \subseteq 2^H$, $I(\mathfrak{B}) = \{I(B) : B \in \mathfrak{B}\}$ is a filterbase on $I(H)$. So that, there exists a filter $\mathfrak{G}$ on $\mathfrak{M}^b(X)$ w^*-convergent to a measure $\nu_0 \in \mathfrak{M}^b(X)$ such that $I(\mathfrak{B}) \subseteq \mathfrak{G}$. From the definition of $t(H)$, there exists a sequence $(K_n)_n \subseteq \mathcal{K}(T)$ such that (1) $|\mu|(T \setminus K_n) < t(H) + 1/n$, for all $n \in \mathbb{N}$, for all $\mu \in H$. We denote $T_0 = \bigcup_{n=1}^{\infty} K_n$ and (2) $X_0 = \bigcup_{n=1}^{\infty} i(K_n) = i(T_0)$.

For every $n \in \mathbb{N}$, $i(K_n) \in \mathcal{K}(X)$, so that X_0 is a Borel set of X. On the other hand, for every $n \in \mathbb{N}$, $X \setminus i(K_n)$ is an open subset of X so that the mapping $\lambda \mapsto |\lambda|(X \setminus i(K_n))$ is a w^*-lower semi-continuous mapping on $\mathfrak{M}^b(X)$ (see [2, Section 5, Proposition 6(a)]).

From $\mathfrak{G} \xrightarrow{w^*} \nu_0$, for all $n \in \mathbb{N}$, there exists $G_n \in \mathfrak{G}$ such that (3) $|\nu_0|(X \setminus i(K_n)) - 1/n < |\lambda|(X \setminus i(K_n))$, for all $\lambda \in G_n$.

The filterbase $\mathfrak{B}$ is a filterbase on H so that $\mathfrak{B} \neq \emptyset$. Let B_0 be a set in $\mathfrak{B}$; then $I(B_0) \in I(\mathfrak{B}) \subseteq \mathfrak{G}$.

For every $n \in \mathbb{N}$, there exists $\mu_n \in B_0$ such that $I(\mu_n) \in G_n$ ($I(B_0) \cap G_n \neq \emptyset$). Therefore, from (1) and (3), for every $n \in \mathbb{N}$,

$$\begin{aligned} |\nu_0|(X \setminus X_0) &\le |\nu_0|\big(X \setminus i(K_n)\big) < \big|I(\mu_n)\big|\big(X \setminus i(K_n)\big) + \frac{1}{n} \\ &= |\mu_n|\Big(i^{-1}\big(X \setminus i(K_n)\big)\Big) + \frac{1}{n} = |\mu_n|(T \setminus K_n) + \frac{1}{n} < t(H) + \frac{2}{n}. \end{aligned} \tag{3.6}$$

Hence (4) $|\nu_0|(X \setminus X_0) \le t(H)$.

Now, X being the Stone-Čech compactification of T, for every $f \in C^b(T)$ there exists $F \in C(X)$ such that $F \circ i = f$ and $\|F\| = \|f\|$ (see [10, Theorem 1.4.6, page 25]). Now we define $J : C^b(T) \to \mathbb{R}$ letting

$$J(f) = \nu_0\big(F \cdot \chi_{X_0}\big) = \int_{X_0} F \, d\nu_0. \tag{3.7}$$

Obviously, J is a continuous linear mapping on $C^b(T)$. For every $\varepsilon > 0$, from (2), there exists $K \in \mathcal{K}(T)$ such that $|\nu_0|(X_0 \setminus i(K)) < \varepsilon$ and $i(K) \subseteq X_0$. Then, for every $g \in C^b(T)$

with $|g| \le 1$ and $g|_K = 0$, let $G \in C(X)$ such that $G \circ i = g$. Therefore, we have

$$\begin{aligned} |J(g)| = \left|\nu_0\Big(G\cdot\chi_{X_0}\Big)\right| &\le \left|\nu_0\Big(G\cdot\chi_{X_0\setminus i(K)}\Big)\right| + \left|\nu_0\Big(G\cdot\chi_{i(K)}\Big)\right| \\ &\le |\nu_0|\big(X_0\setminus i(K)\big) < \varepsilon. \end{aligned} \tag{3.8}$$

Hence, J is a linear mapping satisfying the condition (M) from [2, Section 5, Proposition 5] so that there exists exactly one measure $\mu_0 \in \mathfrak{M}^b(T)$ such that $\mu_0(f) = J(f)$, for every $f \in C^b(T)$. Then we have (5) $\mu_0(f) = \nu_0(F\cdot\chi_{X_0})$, for all $f \in C^b(T)$, where F is the continuous extension of f to X.

Now, for every $f_1,\dots,f_n \in C^b(T)$ with $\|f_k\| > 0$, for all $k = 1,\dots,n$, let $F_1,\dots,F_n \in C(X)$ such that $F_k \circ i = f_k$ and $\|F_k\| = \|f_k\|$, for every $k = 1,\dots,n$.

For all $b > t(H)$, let $\varepsilon = (b - t(H))\cdot\min\{\|f_k\| : k = 1,\dots,n\} > 0$. The set

$$G = \bigcap_{k=1}^{n}\Big\{\lambda \in \mathfrak{M}^b(X) : \big|\lambda(F_k) - \nu_0(F_k)\big| < \varepsilon\Big\} \tag{3.9}$$

is a w^*-neighborhood of ν_0 and so is a member of $\mathfrak{G}$ ($\mathfrak{G} \xrightarrow{w^*} \nu_0$). Therefore, for every $B \in \mathfrak{B}$, $G \cap I(B) \ne \varnothing$ ($I(\mathfrak{B}) \subseteq \mathfrak{G}$). Hence there exists $\mu \in B$ such that $I(\mu) \in G$. Then, for every $k = 1,\dots,n$, from (4) and (5), we have

$$\begin{aligned} \big|\mu(f_k) - \mu_0(f_k)\big| &= \big|\mu(F_k\circ i) - \mu_0(f_k)\big| = \left|I(\mu)(F_k) - \nu_0\Big(F_k\cdot\chi_{X_0}\Big)\right| \\ &\le \big|I(\mu)(F_k) - \nu_0(F_k)\big| + \left|\nu_0\Big(F_k\cdot\chi_{X\setminus X_0}\Big)\right| < \varepsilon + \|F_k\|\cdot|\nu_0|(X\setminus X_0) \\ &< \varepsilon + \|f_k\|\cdot t(H) \le (b - t(H))\cdot\|f_k\| + \|f_k\|\cdot t(H) = b\cdot\|f_k\|. \end{aligned} \tag{3.10}$$

Therefore, $\mu \in \bigcap_{k=1}^{n} V_{b,f_k}(\mu_0)$. So, for every $b > t(H)$, $n \in \mathbb{N}$, $f_1,\dots,f_n \in C^b(T)$ and $B \in \mathfrak{B}$,

$$\bigcap_{k=1}^{n} V_{b,f_k}(\mu_0) \cap B \ne \varnothing. \tag{3.11}$$

Let $\mathfrak{F}$ be the filter generated by the filterbase

$$\left\{\bigcap_{k=1}^{n} V_{b,f_k}(\mu_0) \cap B : b > t(H),\ f_1,\dots,f_n \in C^b(T),\ B \in \mathfrak{B}\right\}. \tag{3.12}$$

Then $\mathfrak{B} \subseteq \mathfrak{F}$ and $\mathfrak{F} \xrightarrow{p_{t(H)}} \mu_0$, so that H is a $t(H)$-relatively compact set. □

Remark 3.9. If H is tight in $\mathfrak{M}^b(T)$ then $t(H) = 0$, so that H is a relatively narrowly compact set and we obtain Prohorov's theorem.

Remark 3.10. Let $a \ge b \ge 0$; then, every b-relatively compact set is a-relatively compact set, also. Therefore, for every bounded set $H \subseteq \mathfrak{M}^b(T)$

$$[t(H), +\infty) \subseteq \{a \ge 0 : H \text{ is } a\text{-relatively compact}\}. \tag{3.13}$$

Remark 3.11. We say that $H \subseteq \mathfrak{M}_+^b(T)$ is *a-relatively compact* in $\mathfrak{M}_+^b(T)$ if, for every filterbase $\mathfrak{B} \subseteq 2^H$, there exist a filter $\mathfrak{F}$ on $\mathfrak{M}_+^b(T)$ and $\mu \in \mathfrak{M}_+^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{F}$ and for all $b > a$, for all $f \in C^b(T), V_{b,f}(\mu) \cap \mathfrak{M}_+^b(T) \in \mathfrak{F}$; we say in this case that $\mathfrak{F} \xrightarrow{p_a} \mu$ in $\mathfrak{M}_+^b(T)$.

The subset of all positive measures, $\mathfrak{M}_+^b(X)$, is closed in the narrow topology of $\mathfrak{M}^b(X)$ (see [2, Section 5, Remark 2]) so that, if $H \subseteq \mathfrak{M}_+^b(T)$ is a bounded subset, then $I(H)$ is w^*-relatively compact in $\mathfrak{M}_+^b(X)$. Then we follow the proof of Theorem 3.8 and we obtain that every bounded subset $H \subseteq \mathfrak{M}_+^b(T)$ is $t(H)$-relatively compact in $\mathfrak{M}_+^b(T)$. Also, we have

$$[t(H), +\infty) \subseteq \left\{a \geq 0 : H \text{ is } a\text{-relatively compact in } \mathfrak{M}_+^b(T)\right\}. \tag{3.14}$$

In the particular case where T is locally compact, we have the converse of Theorem 3.8 in the subspace $\mathfrak{M}_+^b(T)$.

Theorem 3.12. *Let T be a locally compact space and H an a-relatively compact set in $\mathfrak{M}_+^b(T)$; then $t(H) \leq a$.*

Proof. We suppose that H is an a-relatively compact subset of $\mathfrak{M}_+^b(T)$ and $t(H) = \inf_{K \in \mathcal{K}(T)} \sup_{\mu \in H} \mu(T \setminus K) > a$. Then, for every $\varepsilon > 0$ and $K \in \mathcal{K}(T)$, there exists $\mu_K \in H$ such that (1) $\mu_K(T \setminus K) > a + \varepsilon$.

For every $K \in \mathcal{K}(T)$ we denote $B_K = \{\mu_L : L \in \mathcal{K}(T), K \subseteq L\}$. Then $\mathfrak{B} = \{B_K : K \in \mathcal{K}(T)\}$ is a filterbase on H so that there exist a filter $\mathfrak{F}$ on $\mathfrak{M}_+^b(T)$ and $\mu \in \mathfrak{M}_+^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{F}$ and (2) $\mathfrak{F} \xrightarrow{p_a} \mu$, in $\mathfrak{M}_+^b(T)$ (see Remark 3.11). Since μ is a Radon measure, there exists $K_0 \in \mathcal{K}(T)$ such that (3) $\mu(T \setminus K_0) < \varepsilon/2$.

Let U be a relatively compact neighborhood of K and $f : T \to [0,1]$ a continuous function such that (4) $f|_{K_0} = 0$ and $f|_{T \setminus U} = 1$.

We remark that $f \in C^b(T)$ and $\|f\| = 1$. Now let $b = a + \varepsilon/2 > a$ and $f \in C^b(T)$; from (2), $V_{b,f}(\mu) \in \mathfrak{F} \supseteq \mathfrak{B}$ so that (5) $V_{b,f}(\mu) \cap B_{\bar{U}} \neq \emptyset$.

Hence, there exists $K \in \mathcal{K}(T)$, $K \supseteq \bar{U}$ such that (6) $|\mu_K(f) - \mu(f)| \leq b \cdot \|f\| = b$.

From (1), (3), (4), and (6) we obtain the following contradiction:

$$\begin{aligned} a + \varepsilon < \mu_K(T \setminus K) &\leq \mu_K(T \setminus \bar{U}) \leq \mu_K(f) \leq \mu(f) + a + \frac{\varepsilon}{2} \\ &\leq \mu(T \setminus K_0) + a + \frac{\varepsilon}{2} < a + \varepsilon. \end{aligned} \tag{3.15}$$

□

Remark 3.13. If H is a relatively narrowly compact subset of $\mathfrak{M}_+^b(T)$ (i.e., 0-relatively compact set), then $t(H) = 0$ so that H is tight. Therefore, we obtain the converse of Prohorov's theorem; so Theorem 3.12 is an extension of [2, Section 5, Theorem 2].

Remark 3.14. From Remark 3.11 and Theorem 3.12, we obtain (in the case of locally compact spaces)

$$[t(H), +\infty) = \left\{a \geq 0 : H \text{ is } a\text{-relatively compact in } \mathfrak{M}_+^b(T)\right\}. \tag{3.16}$$

Example 3.15. Let $T = \mathbb{N}$ be the set of natural numbers and $\mathfrak{B} = \mathcal{P}(\mathbb{N})$. Then $\mathfrak{M}^b(\mathbb{N}) = \ell^1$ (the space of all sequences of real numbers $(x_n)_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n| < +\infty$) and $C^b(T) = \ell^\infty$ (the space of all bounded sequences of real numbers). Indeed,

$$\begin{gathered} \forall x = (x_n)_n \in \ell^1, \ x : \mathfrak{B} \longrightarrow \mathbb{R}, \quad x(A) = \sum_{n \in A} x_n, \\ x(y) = \sum_n x_n y_n, \quad \forall y = (y_n)_n \in \ell^\infty. \end{gathered} \tag{3.17}$$

Let $(x^p)_{p\in\mathbb{N}} \subseteq \mathfrak{M}^b(\mathbb{N})$ and $x \in \mathfrak{M}^b(\mathbb{N})$, where $x^p = (x^p_n)_n$, for every $p \in \mathbb{N}$ and $x = (x_n)_n$. Then $x^p \xrightarrow{a} x$ if and only if (1) $\limsup_p |\sum_{n\in\mathbb{N}}(x^p_n - x_n)\cdot y_n| \leq a \cdot \sup_n |y_n|$, for all $(y_n)_n \in \ell^\infty$ (see Remark 3.6).

For every bounded set $H = \{x^p : p \in \mathbb{N}\} \subseteq \mathfrak{M}^b(\mathbb{N})$ (2) $t(H) = \inf_m \sup_p \sum_{n=m}^{\infty} |x^p_n|$.

Let $(x_p)_{p\in\mathbb{N}} \subseteq [0,1]$ be a sequence; we define

$$x^p_n = \begin{cases} 1 - x_p, & n = 0, \\ x_p, & n = p, \\ 0, & \text{otherwise}. \end{cases} \tag{3.18}$$

Then $x^p = (x^p_n)_{n\in\mathbb{N}} \in \mathfrak{M}^b(\mathbb{N})$ and, from (2), we obtain

$$t\Big(\{x^p : p \in \mathbb{N}\}\Big) = \limsup_n x_n = t. \tag{3.19}$$

We easily remark that $x^p \xrightarrow{t} x$, where $x = (x_n)_n$ and

$$x_n = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \tag{3.20}$$

From Remark 3.14, $\inf\{a \geq 0 : x^p \xrightarrow{a} x\} = \limsup_n x_n$. If $x_n \to 0$, then $(x^p)_p$ is narrowly convergent to x. In the particular case where $x_n = 1$, for every $n \in \mathbb{N}$, x^p is the Dirac measure δ_p and $\delta_p \xrightarrow{1} \delta_0$.

We remark that

$$\inf\Big\{a \geq 0 : \delta_p \xrightarrow{a} \delta_0\Big\} = 1. \tag{3.21}$$

4. Probabilistic metric on $\mathfrak{M}^1(T)$. Let (T,d) be a Polish space and let $\mathfrak{M}^1(T) \subseteq \mathfrak{M}^b_+(T)$ be the subset of all probabilities on T. We say that a net $(\mu_i)_{i\in I} \subseteq \mathfrak{M}^1(T)$ is *p_a-convergent to $\mu \in \mathfrak{M}^1(T)$ $(a \geq 0)$* if

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a \cdot \|f\|, \quad \forall f \in C^b(T). \tag{4.1}$$

We denote this by $\mu_i \xrightarrow{a} \mu$. So, $\bar{p} = \{p_a : a \in [0,+\infty]\}$ is the narrow limit tower induced on $\mathfrak{M}^1(T)$ (see Remark 3.6). If X is the Stone-Čech compactification of T, the subset $\mathfrak{M}^1(X)$ is a compact set of $\mathfrak{M}^b(X)$ (see [2, Section 5, Proposition 11]). So, with a similar argument to that of Remark 3.11, we deduce that every subset $H \subseteq \mathfrak{M}^1(T)$ is $t(H)$-relatively compact in $\mathfrak{M}^1(T)$ (i.e., every net $(\mu_i)_{i\in I}$ has a subnet p_a-convergent).

Theorem 4.1 has a similar proof to that of Portmanteau's theorem (see [1, Theorem 2.1, Appendix III, Theorem 3]) which we omit.

Theorem 4.1. *Let $(\mu_i)_{i\in I}$ be a net in $\mathfrak{M}^1(T)$, $\mu \in \mathfrak{M}^1(T)$ and $a \geq 0$; the following statements are equivalent:*

$$\mu_i \xrightarrow{a} \mu, \tag{4.2}$$

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a, \quad \forall f \in C^b(T) \text{ with } \|f\| \leq 1, \tag{4.3}$$

$$\limsup_i \mu_i(F) \le \mu(F), \quad \forall F = \bar{F} \subseteq T, \tag{4.4}$$

$$\liminf_i \mu_i(D) \ge \mu(D), \quad \forall D = D^\circ \subseteq T, \tag{4.5}$$

$$\limsup_i \left|\mu_i(A) - \mu(A)\right| \le a, \quad \forall A \in \mathcal{B} \text{ with } \mu(\bar{A} - A^\circ) = 0. \tag{4.6}$$

In Theorem 4.1, $\bar{A}$ and A° denote the closure and the interior of A in the topological space (T, τ_d), respectively.

Remark 4.2. In Theorem 4.1, we can suppose that $a \in [0,1]$.

Remark 4.3. R. Löwen gave a similar result in [7, Theorem 6].

Definition 4.4. For every $F = \bar{F} \subseteq T$ and $\varepsilon > 0$ we denote $F^\varepsilon = \{t \in T : d(t,F) < \varepsilon\}$. For every $a \in [0,1]$ we define $L_a : \mathfrak{M}^1(T) \times \mathfrak{M}^1(T) \to \mathbb{R}_+$ letting

$$L_a(\mu,\nu) = \inf\left\{\varepsilon > 0 : \mu(F) < \nu(F^\varepsilon) + a + \varepsilon, \nu(F) < \mu(F^\varepsilon) + a + \varepsilon, \forall F = \bar{F} \subseteq T\right\}. \tag{4.7}$$

Remark 4.5. L_0 is the metric of Lévy-Prohorov on $\mathfrak{M}^1(T)$. Therefore, L_0 induces the narrow topology on $\mathfrak{M}^1(T)$ and $(\mathfrak{M}^1(T), L_0)$ is a Polish space [2, Section 5, Examples 8 and 9].

Remark 4.6. The family $\mathcal{L} = \{L_a : a \in [0,1]\}$ has the following properties:

$$\begin{gathered}
L_a(\mu,\nu) = 0, \quad \forall a \ge 0 \Longleftrightarrow \mu = \nu, \\
L_a(\mu,\nu) = L_a(\nu,\mu), \quad \forall \mu,\nu \in \mathfrak{M}^1(T), \quad \forall a \in [0,1], \\
L_{a+b}(\mu,\nu) \le L_a(\mu,\lambda) + L_b(\lambda,\nu), \quad \forall \mu,\nu,\lambda \in \mathfrak{M}^1(T), \quad \forall a,b \in [0,1], \\
L_a(\mu,\nu) = \sup_{b>a} L_b(\mu,\nu), \quad \forall \mu,\nu \in \mathfrak{M}^1(T), \quad \forall a \in [0,1).
\end{gathered} \tag{4.8}$$

In [4, Theorem 1] we proved that such a family $\mathcal{L}$ is an equivalent gradated form of a probabilistic metric (F, T_m), where, for every $\mu, \nu \in \mathfrak{M}^1(T)$ and $a > 0$,

$$F(\mu,\nu)(a) = \sup_{\varepsilon>0} \inf_{F=\bar{F}} \left\{\min\left[\mu(F^{a-\varepsilon}) - \nu(F), \nu(F^{a-\varepsilon}) - \mu(F)\right] + 1 + a\right\} \wedge 1 \tag{4.9}$$

and $T_m(a,b) = \max\{a + b - 1, 0\}$. For the space of distribution functions, equivalent probabilistic metrics are introduced in [5, 6, 9].

In Theorem 4.7 we compare the narrow limit tower with the convergence structures induced by the family of semi-pseudometrics $\mathcal{L} = \{L_a : a \in [0,1]\}$. So, this theorem is an important step to obtain a sequential version of Theorem 3.8.

Theorem 4.7. *Let $(\mu_i)_{i\in I}$ be a net in $\mathfrak{M}^1(T)$, $\mu \in \mathfrak{M}^1(T)$ and $a \in [0,1]$.*

$$\text{If } L_a(\mu_i,\mu) \longrightarrow 0, \quad \text{then } \mu_i \xrightarrow{a} \mu, \tag{4.10}$$

$$\text{If } \mu_i \xrightarrow{a} \mu, \quad \text{then } L_{2a}(\mu_i,\mu) \longrightarrow 0. \tag{4.11}$$

Proof. (i) We suppose that $L_a(\mu_i,\mu) \to 0$; then, for every $n \in \mathbb{N}^*$, there exists $i_n \in I$ such that, for every $i \ge i_n$, $L_a(\mu_i,\mu) < 1/n$. Therefore,

$$\mu_i(F) < \mu(F^{1/n}) + a + \frac{1}{n}, \quad \forall F = \bar{F}, \tag{4.12}$$

so that, for every $F = \bar{F} \subseteq T$,

$$\limsup_i \mu_i(F) \le \sup_{i \ge i_n} \mu_i(F) \le \mu(F^{1/n}) + a + \frac{1}{n}. \tag{4.13}$$

But $\mu(F^{1/n}) \to \mu(F)$, so that $\limsup_i \mu_i(F) \le \mu(F) + a$, for all $F = \bar{F}$.

From (4.4) this is equivalent to $\mu_i \xrightarrow{a} \mu$.

(ii) Let now $\mu_i \xrightarrow{a} \mu$ and let $\varepsilon > 0$. For every $r > 0$ and $t \in T$, let $S_r(t) = \{s \in T : d(s,t) < r\}$. Then $\overline{S_r(t)} \setminus S_r^\circ(t) \subseteq \{s \in T : d(s,t) = r\} = C_r$. But $C_{r_1} \cap C_{r_2} = \emptyset$, for all $r_1 \ne r_2$ and $\mu(\cup_{r>0} C_r) \le 1$. It follows that there exists a countable set $N \subseteq (0,+\infty)$ such that $\mu(C_r) = 0$, for all $r \in (0,+\infty) \setminus N$. Therefore, T being separable, there exists a countable family $\{S_{r_n}(t_n) : n \in \mathbb{N}\}$ such that (1) $T = \cup_1^\infty S_{r_n}(t_n)$, $\mu(\overline{S_{r_n}(t_n)} \setminus S_{r_n}^\circ(t_n)) = 0$ and $r_n < \varepsilon/6$, for all $n \in \mathbb{N}$.

We denote for all $n \in \mathbb{N}$, $S_n = S_{r_n}(t_n)$. Let $K \subseteq T$ be a compact set such that $\mu(T \setminus K) < \varepsilon/3$ and let $p \in \mathbb{N}$ such that $K \subseteq \cup_{n=1}^p S_n = A_0$; then (2) $\mu(T \setminus A_0) < \varepsilon/3$.

We denote $\mathcal{A} = \{\cup_{i=1}^q S_{k_i} : q \in \mathbb{N}, k_1, \dots, k_n \le p\}$; obviously, $A_0 \in \mathcal{A}$. For every $A \in \mathcal{A}$, $\mu(\bar{A} \setminus A^\circ) = 0$ so that, from (4.6),

$$\limsup_i |\mu_i(A) - \mu(A)| \le a. \tag{4.14}$$

Therefore there exists $i_0 \in I$ such that, for every $i \ge i_0$ and $A \in \mathcal{A}$, (3) $|\mu_i(A) - \mu(A)| < a + \varepsilon/3$.

Now, for every $F = \bar{F} \subseteq T$, let

$$A_F = \bigcup \left\{ S_n : n \le p,\ S_n \cap F \ne \emptyset \right\} \in \mathcal{A}. \tag{4.15}$$

Then (4) $F \subseteq A_F \cup (T \setminus A_0)$, $A_F \subseteq F^{\varepsilon/3}$.

Indeed, $F = (F \cap A_0) \cup (F \setminus A_0) \subseteq A_F \cup (T \setminus A_0)$. For every $t \in A_F$ there exists S_n such that $t \in S_n$ and $S_n \cap F \ne \emptyset$. Then, from (1), $d(t,F) \le 2 \cdot r_n < \varepsilon/3$, so that $t \in F^{\varepsilon/3}$. Then, from (2), (3), and (4), we have

$$\begin{aligned}
\mu_i(F) &< \mu_i(A_F) + \mu_i(T \setminus A_0) < \mu(F) + a + \frac{\varepsilon}{3} + 1 - \mu_i(A_0) \\
&< \mu(A_F) + a + \frac{\varepsilon}{3} + 1 - \mu(A_0) + a + \frac{\varepsilon}{3} = \mu(A_F) + \mu(T \setminus A_0) + 2 \cdot a + \frac{2\varepsilon}{3} \\
&< \mu(F^{\varepsilon/3}) + 2 \cdot a + \varepsilon \le \mu(F^\varepsilon) + 2 \cdot a + \varepsilon, \\
\mu(F) &\le \mu(A_F) + \mu(T \setminus A_0) < \mu_i(A_F) + a + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \mu_i(F^{\varepsilon/3}) + a + \frac{2\varepsilon}{3} < \mu_i(F^\varepsilon) + 2 \cdot a + \varepsilon,
\end{aligned} \tag{4.16}$$

for every $F = \bar{F} \subseteq T$. Then $L_{2a}(\mu_i, \mu) \le \varepsilon$, for every $i \ge i_0$. Therefore $L_{2a}(\mu_i, \mu) \to 0$. □

Corollary 4.8. *Let H be an a-relatively compact subset in $\mathfrak{M}^1(T)$; then, for every sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq H$, there exist a subsequence $(\mu_{k_n})_{n \in \mathbb{N}}$ and $\mu \in \mathfrak{M}^1(T)$ such that $\mu_{k_n} \xrightarrow{2 \cdot a} \mu$.*

Proof. For every sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq H$, there exist a subnet $(\mu_{n_i})_{i \in I}$ and $\mu \in \mathfrak{M}^1(T)$ such that $\mu_{n_i} \xrightarrow{a} \mu$. From (4.11), $L_{2a}(\mu_{n_i}, \mu) \to 0$. So, for every $p \in \mathbb{N}$, there exists $i_p \in I$ such that $n_{i_p} \ge p$ and $L_{2a}(\mu_{n_{i_p}}, \mu) < 1/p$.

Therefore, we can choose a subsequence $(\mu'_n)_{n\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that $L_{2a}(\mu'_n,\mu)\to 0$. From (4.10) it follows that $\mu'_n \xrightarrow{2\cdot a} \mu$. □

Now we are able to give the sequential version of Theorem 3.8.

Theorem 4.9. *Let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathfrak{M}^1(T)$ and $t = t(\{\mu_n : n\in\mathbb{N}\})$ be the degree of tightness of $(\mu_n)_{n\in\mathbb{N}}$. Then there exist a subsequence $(\mu_{k_n})_{n\in\mathbb{N}}$ and $\mu\in\mathfrak{M}^1(T)$ such that*

$$\mu_{k_n} \xrightarrow{2\cdot t} \mu. \tag{4.17}$$

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