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A NOTE ON THE COMPARISON OF TOPOLOGIES

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ABSTRACT. A considerable problem of some bitopological covering properties is the bitopological unstability with respect to the presence of the pairwise Hausdorff separation axiom. For instance, if the space is RR-pairwise paracompact, its two topologies will collapse and revert to the unitopological case. We introduce a new bitopological separation axiom $\tau S_2\sigma$ which is appropriate for the study of the bitopological collapse. We also show that the property that may cause the collapse is much weaker than some modifications of pairwise paracompactness and we generalize several results of T. G. Raghavan and I. L. Reilly (1977) regarding the comparison of topologies.

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1. Preliminaries. The term space is referred to as a set with one, two, or three topologies, depending on the context. Let X be a set with three topologies τ, σ, and ρ. Recall that X is said to be $(\tau-\sigma)$ paracompact with respect to ρ (see [6]) if every τ-open cover of X has a σ-open refinement which is locally finite with respect to the topology ρ. We say that $x\in X$ is a (σ,ρ)-θ-cluster point (see [4]) of a filter base Φ in X if for every $V\in\sigma$ such that $x\in V$ and every $F\in\Phi$, the intersection $F\cap \mathrm{cl}_\rho V$ is nonempty. If Φ has a cluster point with respect to the topology τ, we say that Φ has a τ-cluster point. Recall that X is called (τ,σ,ρ)-θ-regular if X satisfies any of the following equivalent conditions (see [4]):

(i) For every τ-open cover Ω of X and each $x\in X$ there is a σ-open neighborhood U of x such that $\mathrm{cl}_\rho U$ can be covered by a finite subfamily of Ω.

(ii) Every τ-closed filter base Φ with a (σ,ρ)-θ-cluster point has a τ-cluster point.

(iii) Every filter base Φ with a (σ,ρ)-θ-cluster point has a τ-cluster point.

(iv) For every filter base Φ in X with no τ-cluster point and every $x\in X$ there are $U\in\sigma$, $V\in\rho$, and $F\in\Phi$ such that $x\in U$, $F\subseteq V$, and $U\cap V=\varnothing$.

For example, a space which is $(\tau-\rho)$ paracompact with respect to σ is (τ,σ,ρ)-θ-regular. Let (X,τ,σ) be a bitopological space. Recall that X is said to be τR_0 if the topology τ is R_0, that is, if $x\in U\in\tau$ implies $\mathrm{cl}_\tau\{x\}\subseteq U$. We say that the space X is $\tau R\sigma$ (see [6]) if for each $x\in X$, $\mathrm{cl}_\tau\{x\}=\bigcap\{V\mid V\in\sigma,\ x\in V\}=\bigcap\{\mathrm{cl}_\tau V\mid V\in\sigma,\ x\in V\}$. Recall that the space X is called τ locally compact with respect to σ (see [5]) if each point $x\in X$ has a neighborhood in σ, which is compact in τ. We say that X is τ paracompact with respect to σ (see [5]) if each τ-open cover of X has a τ-open refinement which is locally finite with respect to σ, that is, if X is $(\tau-\tau)$ paracompact with respect to σ. The space X is said to be τ α-paracompact with respect to σ (see [6]) if every τ-open cover of X admits a $(\tau\vee\sigma)$-open refinement which is locally finite with respect to σ. The bitopological space X is said to be α-pairwise paracompact (see [6]) if X is $(\tau-(\tau\vee\sigma))$ paracompact with respect to σ and $(\sigma-(\tau\vee\sigma))$ paracompact with respect to τ. The space X is called RR-pairwise paracompact (see [6]) if X is $(\tau-\tau)$ paracompact with respect to σ and $(\sigma-\sigma)$ paracompact with respect to τ. Finally, we say that X is FHP-pairwise paracompact (see [2]) if $X(\tau-\sigma)$ paracompact with respect to σ and $(\sigma-\tau)$ paracompact with respect to τ. Note that most of the previous bitopological notions were introduced by T. G. Raghavan and I. L. Reilly [5], with one exception—the concept of FHP-pairwise paracompactness is due to P. Fletcher, H. B. Hoyle III, and C. W. Patty [2].

2. Main results. I. L. Reilly and T. G. Raghavan in [6] used $\tau R\sigma$ axiom in the conjunction with τR_0 to replace pairwise Hausdorff axiom in some comparative and coincidence theorems for the topologies of bitopological spaces which satisfy some modifications of pairwise paracompactness. However, some relatively large classes of topological spaces were unfortunately excluded from Raghavan-Reilly's results. The main occasion was that pairwise paracompactness is too strong and global property for characterizing the essential reasons of the bitopological collapse in a sufficiently soft setting. The second occasion was that $\tau R\sigma$ axiom is also rather strong—it contains the condition $\bigcap\{V \mid V\in\sigma,\ x\in V\}=\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$ which is useless in this context. We introduce a new concept that is a generalization of the conjunction of $\tau R\sigma$ and τR_0 and is more appropriate for studying the bitopological collapse. Our new axiom is inspired by S_2 axiom of [1] (which is also equivalent to the well-known R_1—see [3] for the definition).

Definition 2.1. Let (X,τ,σ) be a bitopological space. We say that X is $\tau S_2\sigma$ if for every $x,y\in X$ and for each $U\in\tau$ with $x\in U$, $y\notin U$ there exist $V\in\sigma$, $W\in\tau$ such that $x\in V$, $y\in W$, and $V\cap W=\emptyset$.

Definition 2.2. Let (X,τ,σ) be a bitopological space. We say that X is pairwise S_2 if X is $\tau S_2\sigma$ and $\sigma S_2\tau$.

Proposition 2.3. *A bitopological space (X,τ,σ) is $\tau S_2\sigma$ if and only if X is τR_0 and* $\mathrm{cl}_\tau\{x\}=\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$ *for every $x\in X$.*

Proof. Suppose that X is $\tau S_2\sigma$ and let $y\in U=X\setminus\mathrm{cl}_\tau\{x\}$. Since X is $\tau S_2\sigma$, there exist $V\in\sigma$, $W\in\tau$ such that $y\in V$, $x\in W$, and $V\cap W=\emptyset$. Then $y\notin W$. Applying the $\tau S_2\sigma$ axiom again, we get some $P\in\sigma$, $Q\in\tau$ such that $x\in P$, $y\in Q$, and $P\cap Q=\emptyset$. Then $\mathrm{cl}_\tau P\subseteq X\setminus Q$ which implies that $y\notin\mathrm{cl}_\tau P$. Hence, $y\in X\setminus\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$. Therefore, $\mathrm{cl}_\tau\{x\}=\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$.

Now, let $U\in\tau$ be a τ-neighborhood of some $x\in X$. Since X is $\tau S_2\sigma$, for any $y\in X\setminus U$, there exist $V\in\sigma$, $W\in\tau$ such that $x\in V$, $y\in W$, and $V\cap W=\emptyset$. Then $y\notin\mathrm{cl}_\tau\{x\}$, which implies that $\mathrm{cl}_\tau\{x\}\subseteq U$. Hence X is τR_0.

Conversely, suppose that X is τR_0 and $\mathrm{cl}_\tau\{x\}=\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$ for every $x\in X$. Let $x\in U$ and $U\in\tau$. Then $\mathrm{cl}_\tau\{x\}=\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}\subseteq U$, which implies, for any $y\in X\setminus U$, that $y\notin\bigcap\{\mathrm{cl}_\tau V \mid V\in\sigma,\ x\in V\}$. Then there exist $V\in\sigma$, $W\in\tau$ such that $x\in V$, $y\in W$, and $V\cap W=\emptyset$. Hence X is $\tau S_2\sigma$. □

Corollary 2.4. *Let (X,τ,σ) be a τR_0 and $\tau R\sigma$ space. Then (X,τ,σ) is a $\tau S_2\sigma$ space.*

The following two simple examples show that $\tau S_2\sigma$ and $\tau R\sigma$ (without τR_0) are independent bitopological properties. We leave it to the reader to verify that the constructed spaces have the requested features.

Example 2.5. Let $X = \{1,2,3\}$. Let τ be a topology on X with the base $\tau_0 = \{\{1,2\},\{3\}\}$ and let σ be the discrete topology on X. Then (X,τ,σ) is a bitopological space which is $\tau S_2\sigma$ but not $\tau R\sigma$.

Example 2.6. Let $X = \{0,1\}$ and let $\tau = \{\varnothing,\{0\},X\}$, $\sigma = \{\varnothing,\{1\},X\}$ be the dual Sierpiński topologies on X. Then (X,τ,σ) is a bitopological space which is $\tau R\sigma$ but not $\tau S_2\sigma$.

It is a natural question whether there exists a $\tau S_2\sigma$ non-$\tau R\sigma$ space with the topology τ satisfying at least T_0 axiom. Proposition 2.7 answers this question in negative. However, in this case both properties coincide with pairwise Hausdorff, so the negative answer does not disqualify $\tau S_2\sigma$ relative to $\tau R\sigma$ in conjunction with τR_0 as an irrelevant generalization.

Proposition 2.7. *Let (X,τ,σ) be a bitopological space with τ satisfying the T_0 axiom. The following statements are equivalent:*

(i) *X is pairwise Hausdorff.*
(ii) *X is τR_0 and $\tau R\sigma$.*
(iii) *X is $\tau S_2\sigma$.*

Proof. The implications (i)⇒(ii)⇒(iii) are clear. Suppose that X is $\tau S_2\sigma$. Let $x,y \in X$. Since τ is T_0 topology, we can assume, without loss of generality, that there exist $U \in \tau$ with $x \in U$, $y \notin U$. By Definition 2.1, there exist $V \in \sigma$, $W \in \tau$ such that $x \in V$, $y \in W$, and $V \cap W = \varnothing$. In particular, $x \notin W$. To complete the proof, it is sufficient to exchange the role of x,y and repeat the previous consideration. Hence X is pairwise Hausdorff. □

Therefore, both $\tau S_2\sigma$ and $\tau R\sigma$ (in a conjunction with τR_0) are significant for spaces (X,τ,σ) with non-T_0 topologies τ only.

Corollary 2.8. *Let (X,τ,σ) be a bitopological space. Suppose that at least one of the topologies τ, σ is T_0. Then X is pairwise S_2 if and only if X is pairwise Hausdorff.*

Theorem 2.9. *Let τ,σ,ρ be topologies on X. Suppose that X is $\tau S_2\sigma$ and (τ,σ,ρ)-θ-regular. Then $\tau \subseteq \sigma$.*

Proof. Let $x \in U \in \tau$. Then for every $y \in X \setminus U$ there exist $V_y \in \sigma$ and $U_y \in \tau$ such that $x \in V_y$, $y \in U_y$, and $V_y \cap U_y = \varnothing$. Then $\Omega = \{U\} \cup \{U_y \mid y \in X \setminus U\}$ is a τ-open cover of X. Hence, there exist $V \in \sigma$ and $y_1,y_2,\dots,y_k \in X \setminus U$ such that $\mathrm{cl}_\rho V \subseteq U \cup (\bigcup_{i=1}^k U_{y_i})$. Let $W = V \cap (\bigcap_{i=1}^k V_{y_i})$. Clearly, $x \in W \in \sigma$, $W \subseteq U \cup (\bigcup_{i=1}^k U_{y_i})$, and $W \cap U_{y_i} = \varnothing$ for $i = 1,2,\dots,k$. Then $x \in W \subseteq U$ which implies that $U \in \sigma$. Hence $\tau \subseteq \sigma$. □

Remark 2.10. Note that, one can take any topology on X as ρ. For example, $\rho = 1_X$, where 1_X is the discrete topology on X.

Remark 2.11. The space constructed in Example 2.6 is $\tau R\sigma$ and (τ,σ,ρ)-θ-regular, where ρ is any topology on X, but $\tau \nsubseteq \sigma$. Hence $\tau R\sigma$, without τR_0, cannot replace $\tau S_2\sigma$ in Theorem 2.9.

The following result is independently (see [5]) due to P. Fletcher et al. [2], R. A. Stoltenberg [7], and J. D. Weston [8].

Corollary 2.12 (see [2, 7, 8]). *Let (X,τ,σ) be pairwise Hausdorff and τ be compact. Then $\tau \subseteq \sigma$.*

Taking $\tau S_2\sigma$ instead of pairwise Hausdorff and τ locally compact with respect to σ instead of τ compact, the previous result can be improved.

Corollary 2.13. *Suppose that (X,τ,σ) is $\tau S_2\sigma$ and τ locally compact with respect to σ. Then $\tau \subseteq \sigma$.*

Proof. If X is τ locally compact with respect to σ, then for every $x \in X$ there is τ-compact $H \subseteq X$ with $x \in \operatorname{int}_\sigma H$. Then $U = \operatorname{int}_\sigma H$ is a σ-open neighborhood of x which can be covered by a finite subfamily of any τ-open cover of X. Hence, X is $(\tau,\sigma,1_X)$-θ-regular and $\tau S_2\sigma$, so $\tau \subseteq \sigma$ by Theorem 2.9. $\square$

Corollary 2.14 (see [5]). *If (X,τ,σ) is pairwise regular, pairwise Hausdorff and τ locally compact with respect to σ, then $\tau \subseteq \sigma$.*

Regarding bitopological paracompactness, we have the following corollary of Theorem 2.9.

Corollary 2.15. *Let (X,τ,σ) be $\tau S_2\sigma$ and suppose that every τ-open cover of X has a refinement which is locally finite with respect to σ. Then $\tau \subseteq \sigma$.*

Proof. Since X is $(\tau,\sigma,1_X)$-θ-regular by the definition and $\tau S_2\sigma$, the corollary now follows from Theorem 2.9. $\square$

Corollary 2.16. *Let (X,τ,σ) be $\tau S_2\sigma$ and τ α-paracompact with respect to σ. Then $\tau \subseteq \sigma$.*

Corollary 2.17 (see [6]). *If (X,τ,σ) is $\tau R_0, \tau R\sigma$, and τ α-paracompact with respect to σ, then $\tau \subseteq \sigma$.*

Corollary 2.18 (see [5]). *If (X,τ,σ) is pairwise Hausdorff and τ paracompact with respect to σ, then $\tau \subseteq \sigma$.*

Corollary 2.19. *Let (X,τ,σ) be $\tau S_2\sigma$ and $(\tau'-\rho')$ paracompact with respect to σ'. If $\tau \subseteq \tau'$ and $\sigma' \subseteq \sigma$, then $\tau \subseteq \sigma$.*

The following three originally independent results of [6] now immediately follow from Corollary 2.19. We repeat them because of completeness.

Corollary 2.20 (see [6]). *If (X,τ,σ) is $\tau R_0, \tau R\sigma$, and $(\tau'-\rho')$ paracompact with respect to σ', then $\tau \subseteq \sigma$ if $\tau \subseteq \tau' \subseteq \tau \vee \sigma$ and $\tau \wedge \sigma \subseteq \sigma' \subseteq \sigma$.*

Corollary 2.21 (see [6]). *Let (X,τ,σ) be pairwise Hausdorff and $(\tau'-\rho')$ paracompact with respect to σ'. Then $\tau \subseteq \sigma$ if $\tau \subseteq \tau' \subseteq \tau \vee \sigma$ and $\tau \wedge \sigma \subseteq \sigma' \subseteq \sigma$.*

Corollary 2.22 (see [6]). *Let (X, τ, σ) be pairwise Hausdorff and $(\tau' - \rho')$ paracompact with respect to σ'. Then $\tau \subseteq \sigma$ if $\tau \subseteq \tau' \subseteq \tau \vee \sigma$ and $\tau \wedge \sigma \subseteq \rho'$, $\sigma' \subseteq \sigma$.*

Definition 2.23. Let (X, τ, σ) be a bitopological space. We say that the topologies τ and σ are K-equivalent if X is $(\tau, \sigma, 1_X)$-θ-regular and $(\sigma, \tau, 1_X)$-θ-regular.

Remark 2.24. Note that, from the definition of $(\tau, \sigma, 1_X)$-θ-regularity it follows that two topologies on a set X are K-equivalent if and only if any filter base in X which has a cluster point in one topology has a (not necessarily the same) cluster point in the other topology; or equivalently, if for every cover of X open in one topology and every point $x \in X$ there is a neighborhood of x open in the other topology which can be covered by finitely many members of the cover. Of course, the property of K-equivalence is an equivalence relation on the set of all topologies on X.

Corollary 2.25. *Let (X, τ, σ) be pairwise S_2. Then $\tau = \sigma$ if and only if the topologies τ, σ are K-equivalent.*

According to Remark 2.11, one can easily check that α-pairwise paracompact and then also RR-pairwise paracompact and FHP-pairwise paracompact bitopological spaces have K-equivalent topologies. Hence these modifications of bitopological paracompactness will revert to the unitopological setting in presence of pairwise S_2 separation axiom.

Corollary 2.26 (see [6]). *Let (X, τ, σ) be pairwise Hausdorff and α-pairwise paracompact. Then $\tau = \sigma$.*

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