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STABLE RINGS GENERATED BY THEIR UNITS

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Abstract. We introduce the class of rings satisfying $(m,1)$-stable range and investigate equivalent characterizations of such rings. These give generalizations of the corresponding results by Badawi (1994), Ehrlich (1976), and Fisher and Snider (1976).

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Let R be an associative ring with identity. A ring R is said to have stable range one provided that $aR+bR=R$ implies that $a+by\in U(R)$ for $y\in R$. It is well known that M_R cancels from direct sums if $\text{End}_R M$ has stable range one. For further properties of stable range one condition, we refer the reader to [1, 2, 5, 7, 9, 10, 13, 14].

Many authors have studied rings generated by their units (see [3, 4, 7, 8, 10, 12]). It was shown that every unit-regular ring in which 2 is invertible is generated by its unit (see [7, Theorem 5]) and every strongly π-regular ring in which 2 is invertible is generated by its units (see [8, Theorem 3]). So far one always investigate such rings under stable range one condition.

In this paper, we generalize stable range one condition and introduce rings satisfying $(m,1)$-stable range so as to investigate rings generated by their units. Also we give generalizations of the corresponding results in [3, 7, 8].

Throughout, rings are associative with identity and modules are right modules. $\text{GL}_n(R)$ denotes the general linear group of R, $U(R)$ denotes the set of units of R, and that $U_m(R)=\{x\in R\mid \exists u_1,\dots,u_m\in U(R)$ such that $x=u_1+\cdots+u_m\}$. Let $B_{ij}(x)=I_2+xe_{ij}$ $(i\neq j,\ 1\leq i,\ j\leq 2)$, $[\alpha,\beta]=\alpha e_{11}+\beta e_{22}$, where e_{ij} $(1\leq i,\ j\leq 2)$ are matrix units (1 in the i,j position and 0 elsewhere).

Definition 1. The ring R is said to satisfy $(m,1)$-stable range provided that $aR+bR=R$ implies that $a+by\in U(R)$ for $y\in U_m(R)$.

Proposition 2. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *Whenever $ax+b=1$, there exists $y\in U_m(R)$ such that $a+by\in U(R)$.*

Proof. (1)⇒(2). The proof is obvious.

(2)⇒(1). Given $aR+bR=R$, then $ax+by=1$ for some $x,y\in R$. So we can find $z\in U_m(R)$ such that $axz+b=u\in U(R)$, and then $axzu^{-1}+bu^{-1}=1$. Hence we have $w\in U_m(R)$ such that $a+bu^{-1}w\in U(R)$. Clearly, $u^{-1}w\in U_m(R)$, as desired. □

Proposition 3. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *The ring $R/J(R)$ satisfies $(m,1)$-stable range.*

Proof. (1)⇒(2). Given $\bar{a}\bar{x}+\bar{b}=\bar{1}$ in $R/J(R)$, then $ax+(b+r)=1$ for some $r\in J(R)$. Since R satisfies $(m,1)$-stable range, we have $y\in U_m(R)$ such that $a+(b+r)y\in U(R)$. Therefore $\bar{a}+\bar{b}\bar{y}\in U(R/J(R))$ with $\bar{y}\in U_m(R/J(R))$, hence $R/J(R)$ satisfies $(m,1)$-stable range by Proposition 2.

(2)⇒(1). Given $ax+b=1$ in R, then $\bar{a}\bar{x}+\bar{b}=\bar{1}$ in $R/J(R)$. So there is $\bar{y}\in U_m(R/J(R))$ such that $\bar{a}+\bar{b}\bar{y}=\bar{u}\in U(R/J(R))$. Assume that $y=w_1+w_2+\cdots+w_m$ with all $\overline{w_i}\in U(R/J(R))$. Since units lift modulo $J(R)$, we may assume that all $w_i\in U(R)$ and $u\in U(R)$, and that $a+b(w_1+w_2+\cdots+w_m)=u+r$ for some $r\in J(R)$. Obviously, $u+r\in U(R)$ and $w_1+w_2+\cdots+w_m\in U_m(R)$. Hence R satisfies $(m,1)$-stable range, as asserted. □

Theorem 4. *Let R be an associative ring with identity, K a set of some elements of R. Then the following are equivalent:*

(1) *Whenever $ax+b=1$, there exists $y\in K$ such that $a+by\in U(R)$.*

(2) *Whenever $ax+b=1$, there exists $z\in K$ such that $x+zb\in U(R)$.*

Proof. (1)⇒(2). Since $ax+b=1$, we see that $\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}^{-1}=\begin{pmatrix} x & 1-xa \\ -1 & a \end{pmatrix}\in \mathrm{GL}_2(R)$. Clearly, $xa+(1-xa)=1$. So there exists $z\in K$ such that $x+(1-xa)z=u\in U(R)$. Hence $\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}=\begin{pmatrix} u & * \\ * & * \end{pmatrix}\in \mathrm{GL}_2(R)$. Thus we know that $\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}^{-1}=\begin{pmatrix} u & * \\ * & * \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$. Therefore $\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} u & * \\ * & * \end{pmatrix}^{-1}$. Since there is $w\in U(R)$ such that $\begin{pmatrix} u & * \\ * & * \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, we have $v=w^{-1}\in U(R)$ such that $\begin{pmatrix} u & * \\ * & * \end{pmatrix}^{-1}=\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u^{-1} & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$. Hence $\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\begin{pmatrix} * & * \\ * & v \end{pmatrix}$, $v\in U(R)$. So $\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\begin{pmatrix} a & -b \\ 1 & x \end{pmatrix}=\begin{pmatrix} * & * \\ * & v \end{pmatrix}$. Thus, we see that $x+zb=v\in U(R)$, as required.

(2)⇒(1). Applying (1)⇒(2) to the opposite ring R^{op}, we complete the proof. □

Theorem 4 is a general result for symmetry of stable range conditions. As applications, we see that stable range one conditions, unit 1-stable range conditions and rings having many unit-regular elements are symmetric. The following result shows that $(m,1)$-stable range condition is right-left symmetric.

Corollary 5. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *Whenever $ax+b=1$, there exists some $z\in U_m(R)$ such that $x+zb\in U(R)$.*

(3) *Whenever $Ra+Rb=R$, there exists some $z\in U_m(R)$ such that $a+zb\in U(R)$.*

Proof. (1)⇔(2). Set $K=U_m(R)$. Then the equivalence follows by Theorem 4.

(3)⇒(2). The proof is trivial.

(2)⇒(3). Given $Ra+Rb=R$, then $xa+yb=1$ for some $x,y\in R$. So we have $s\in U_m(R)$ such that $sxa+b=u\in U(R)$, hence $u^{-1}sxa+u^{-1}b=1$. Therefore $a+vu^{-1}b\in U(R)$ for some $v\in U_m(R)$, as required. □

Proposition 6. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *For any $A\in \mathrm{GL}_2(R)$, there exists some $w\in U_m(R)$ such that $A=[*,*]B_{21}(w)B_{12}(*)B_{21}(*)$.*

(3) *For any $A\in \mathrm{GL}_2(R)$, there exists some $w\in U_m(R)$ such that $A=[*,*]B_{12}(*)B_{21}(*)B_{12}(w)$.*

Proof. (1)⇒(2). Let $A \in \mathrm{GL}_2(R)$, and let $A^{-1} = (b_{ij})$. Since $b_{11}R + b_{12}R = R$, we can find some $y \in U_m(R)$ such that $b_{11} + b_{12}y = u \in U(R)$. We easily check that $A^{-1} = B_{21}(b_{21} + b_{22}u^{-1})[u, b_{22} - (b_{21} + b_{22}y)u^{-1}b_{12}]B_{12}(u^{-1}b_{12})B_{21}(-y)$. Thus $A = [*,*]B_{21}(w)B_{12}(*)B_{21}(*)$ for some $w \in U_m(R)$.

(2)⇒(1). Given $ax + b = 1$ in R, then we have $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$. Thus we have a $w \in U_m(R)$ such that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right)^{-1} = [*,*]B_{21}(w)B_{12}(*)B_{21}(*)$. Therefore we see that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [*,*]B_{21}(*)B_{12}(*)B_{21}(-y)$ for some $y \in U_m(R)$. Consequently, $a + by \in U(R)$ with $y \in U_m(R)$, as desired.

(1)⇔(3). Applying (1)⇔(2) to the opposite ring R^{op}, we complete the proof by the symmetry of $(m,1)$-stable range conditions. □

Let R be generated by m units. If R has stable range one, then it satisfies $(m,1)$-stable range. Conversely, we easily check that every ring satisfying $(m,1)$-stable range is generated by $m+1$ units. Now we show that $(m,1)$-stable range condition is inherited by matrix rings.

Lemma 7. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *Given $ax + b = 1$ in R, then there exists $y \in R$ such that $a + by \in U(R)$ and $1 - xy \in U_m(R)$.*

(3) *Given $ax + b = 1$ in R, then there exists $z \in R$ such that $x + zb \in U(R)$ and $1 - za \in U_m(R)$.*

Proof. (1)⇒(2). Given $ax + b = 1$ in R, then $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$. In view of Proposition 6, we have a $w \in U_m(R)$ such that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [*,*]B_{21}(w)B_{12}(*)B_{21}(*)$. So we can find some $-y \in R$ such that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [*,*]B_{21}(w)B_{12}(*)B_{21}(-y)$. Therefore $a + by \in U(R)$ and $1 - xy = -(-1 + xy) \in U_m(R)$, as required.

(2)⇒(1). Given $ax + b = 1$ in R, then there exists some $y \in R$ such that $a + by = u \in U(R)$ and $1 - xy = v \in U_m(R)$. So we know that $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} u & b \\ -v & x \end{smallmatrix}\right) = [*,*]B_{21}(w)B_{12}(*)$ for some $w \in U_m(R)$. Thus $\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [*,*]B_{21}(w)B_{12}(*)B_{21}(-y)$. So we can find $z \in U_m(R)$ such that $\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ -1 & x \end{smallmatrix}\right) = [*,*]B_{12}(*)B_{21}(*)$. Consequently, we show that $x + zb \in U(R)$ for some $z \in U_m(R)$. Therefore R satisfies $(m,1)$-stable range by Corollary 5.

(1)⇔(3). Applying (1)⇔(2) to the opposite ring R^{op}, we complete the proof. □

In [6], the author shows that every matrix ring over a ring satisfying unit 1-stable range also satisfies unit 1-stable range. Now we extend [6, Theorem 2.2] to $(m,1)$-stable range conditions by a similar route.

Theorem 8. *If R satisfies $(m,1)$-stable range, then so does $M_n(R)$ for any $n \geq 1$.*

Proof. Given $BC + D = I_n$ in $M_n(R)$, then $A = \left(\begin{smallmatrix} B & D \\ -I_n & C \end{smallmatrix}\right) \in \mathrm{GL}_{2n}(R)$. Set $A = (\mathbb{A}_{ij})$ $(1 \leq i,j \leq 2)$ with all $\mathbb{A}_{ij} = (a_{st}^{ij}) \in M_n(R)$ $(1 \leq s,\ t \leq n)$. Then there exist $x_1, \dots, x_n, y_1, \dots, y_n \in R$ such that $a_{11}^{11}x_1 + \cdots + a_{1n}^{11}x_n + a_{11}^{12}y_1 + \cdots + a_{1n}^{12}y_n = 1, \dots, a_{n1}^{11}x_1 + \cdots + a_{nn}^{11}x_n + a_{n1}^{12}y_1 + \cdots + a_{nn}^{12}y_n = 0$, $a_{11}^{21}x_1 + \cdots + a_{1n}^{21}x_n + a_{11}^{22}y_1 + \cdots + a_{1n}^{22}y_n = 0, \dots, a_{n1}^{21}x_1 + \cdots + a_{nn}^{21}x_n + a_{n1}^{22}y_1 + \cdots + a_{nn}^{22}y_n = 0$. In view of Lemma 7, there is $z_1 \in R$ such that $a_{11}^{11} + a_{12}^{11}x_2z_1 + \cdots + a_{1n}^{11}x_nz_1 + a_{11}^{12}y_1z_1 + \cdots + a_{1n}^{12}y_nz_1 = u_1 \in U(R)$ and $1 - x_1z_1 = v_1 \in U_m(R)$. So we claim that

$$[*,*]A[*,*]B_{21}(*) = \begin{pmatrix} u_1 & a_{12}^{11} & \cdots & a_{1n}^{11} & a_{11}^{12} & \cdots & a_{1n}^{12} \\ 0 & b_{22}^{11} & \cdots & b_{2n}^{11} & b_{21}^{12} & \cdots & b_{2n}^{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2}^{11} & \cdots & b_{nn}^{11} & b_{n1}^{12} & \cdots & b_{nn}^{12} \\ a_{11}^{21}v_1 & a_{12}^{21} & \cdots & a_{1n}^{21} & a_{11}^{22} & \cdots & a_{1n}^{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{21}v_1 & a_{n2}^{21} & \cdots & a_{nn}^{21} & a_{n1}^{22} & \cdots & a_{nn}^{22} \end{pmatrix}. \tag{1}$$

Likewise, we have $u_2,u_3,\dots,u_n \in U(R)$ and $v_2,v_3,\dots,v_n \in U_m(R)$ such that

$$[*,*]A[*,*]B_{21}(*) = \begin{pmatrix} u_1 & * & * & \cdots & * & a_{11}^{12} & \cdots & a_{1n}^{12} \\ 0 & u_2 & * & \cdots & * & b_{21}^{12} & \cdots & b_{2n}^{12} \\ 0 & 0 & u_3 & \cdots & * & c_{31}^{12} & \cdots & c_{3n}^{12} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_n & d_{n1}^{12} & \cdots & d_{nn}^{12} \\ a_{11}^{21}v_1 & a_{12}^{21}v_2 & a_{13}^{21}v_3 & \cdots & a_{1n}^{21}v_n & a_{11}^{22} & \cdots & a_{1n}^{22} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{21}v_1 & a_{n2}^{21}v_2 & a_{n3}^{21}v_3 & \cdots & a_{nn}^{21}v_n & a_{n1}^{22} & \cdots & a_{nn}^{22} \end{pmatrix}. \tag{2}$$

Similar to the consideration in [6, Theorem 2.2] , we can find some $E \in \mathrm{GL}_n(R)$ such that $[*,*]A[*,*]B_{21}(*) = [*,*]B_{21}(-E^{-1}\operatorname{diag}(v_1,\dots,v_n))B_{12}(*)$. Consequently, $A = [*,*]B_{21}(W)B_{12}(*)B_{21}(*)$ with $W \in U_m(M_n(R))$. So there is $W' \in U_m(M_n(R))$ such that

$$B_{21}(W')\begin{pmatrix} B & D \\ -I_n & C \end{pmatrix} = [*,*]B_{12}(*)B_{21}(*), \quad \text{so } C+W'D \in \mathrm{GL}_n(R). \tag{3}$$

It follows from Corollary 5 that $M_n(R)$ satisfies $(m,1)$-stable range. □

Corollary 9. *Let R satisfy $(m,1)$-stable range, then every $n\times n$ matrix over R is the sum of $m+1$ invertible matrices.*

Proof. Let $A \in M_n(R)$. Since R satisfies $(m,1)$-stable range, so does $M_n(R)$ from Theorem 8. As $AM_n(R)+I_nM_n(R) = M_n(R)$, we can find some $U \in U_m(M_n(R))$ such that $A+I_n\times U = V \in \mathrm{GL}_n(R)$. Thus $A = (-U)+V$, as desired. □

Recall that a ring R is said to be an exchange ring if for every right R-module A and any two decompositions $A = M'\oplus N = \bigoplus_{i\in I}A_i$, where $M'_R \cong R_R$ and the index set I is finite, then there exist submodules $A'_i \subseteq A_i$ such that $A = M'\oplus(\bigoplus_{i\in I}A'_i)$. A ring R is said to be strongly π-regular provided that for any $x\in R$, there exists a positive integer n such that $x^n = x^{n+1}y$ for some $y\in R$.

We note that R satisfies $(m,1)$-stable range if and only if it has stable range one and for any $x,y\in R$, there exists $w\in U_m(R)$ such that $xy+xw+1\in U(R)$. By an argument of M. Henriksen [11], we claim that the ring R has stable range one if and only if the ring $M_2(R)$ satisfies $(3,1)$-stable range. For exchange rings, we now derive the following.

Lemma 10. *Let R be an exchange ring with $1/2\in R$. Then the following are equivalent:*

(1) *The exchange ring R has stable range one.*

(2) *The exchange ring R satisfies $(7,1)$-stable range.*

Proof. (2)⇒(1). The proof is clear.

(1)⇒(2). Given $ax+b=1$ in R, then $a+by\in U(R)$ for $y\in R$. Since R is an exchange ring, there exists an idempotent $e\in R$ such that $e=ys$ and $1-e=(1-y)t$. Obviously, ey and $(1-e)(1-y)$ are both regular. Thus $ey=fu,(1-e)(1-y)=gv$ for some $f=f^2$, $g=g^2\in R$ and $u,v\in U(R)$. Hence $y=ey-(1-e)(1-y)+1-e=fu-gv+1-e$. As $2\in U(R)$, we see that $f=2^{-1}+2^{-1}(2f-1)$, $g=2^{-1}+2^{-1}(2g-1)$ and $e=2^{-1}+2^{-1}(2e-1)$. Clearly, $2^{-1}(2f-1),2^{-1}(2g-1),2^{-1}(2e-1)\in U(R)$. Therefore $y\in U_7(R)$, as required. □

Theorem 11. *Let R be a strongly π-regular ring. If 2 is a nonnilpotent of R, then there exists some nonzero idempotent $e\in R$ such that $M_n(eRe)$ satisfies $(7,1)$-stable range.*

Proof. Since R is a strongly π-regular ring, there exists $n\geq 1$ such that $2^n=eu$ for some $e=e^2$, $u\in U(R)$. Since 2 is a nonnilpotent of R, we see that $e\neq 0$. Assume that $uv=1$ for $v\in R$. We easily check that $(eue)(eve)=2^neve=euve=e$. Likewise, we have $(eve)(eue)=e$. Thus $2e\in U(eRe)$. On the other hand, we know that eRe is a strongly π-regular ring. By virtue of [1, Theorem 4], R has stable range one. Thus we complete the proof by Theorem 8 and Lemma 10. □

Proposition 12. *The following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *Whenever $aR+bR=dR$, there exist $y\in U_m(R)$, $u\in U(R)$ such that $a+by=du$.*

(3) *Whenever $Ra+Rb=dR$, there exist $z\in U_m(R)$, $u\in U(R)$ such that $a+zb=ud$.*

Proof. (1)⇒(2). Given $aR+bR=dR$, then $(a,b)M_2(R)=(d,0)M_2(R)$. Assume that $(d,0)A=(a,b)$ and $(a,b)B=(d,0)$. From $AB+(I_2-AB)=I_2$, we have $Y\in M_2(R)$ such that $A+(I_2-AB)Y=W\in \mathrm{GL}_2(R)$. Thus $(a,b)=(d,0)A=(d,0)(A+(I_2-AB))=(d,0)W$. Assume that $W=(w_{ij})$. Then $w_{11}R+w_{12}R=R$, whence $w_{11}+w_{12}y=u\in U(R)$ for $y\in U_m(R)$. Therefore $a+by=du$, as desired.

(2)⇒(1). The proof is trivial.

(1)⇔(3). Applying (1)⇔(2) to the opposite ring R^{op}, we complete the proof by the symmetry of $(m,1)$-stable range property. □

Corollary 13. *Let R be a ring which is quasi-injective as a right R-module. Then the following are equivalent:*

(1) *The ring R satisfies $(m,1)$-stable range.*

(2) *Whenever $r\cdot\mathrm{ann}(a)\cap r\cdot\mathrm{ann}(b)=r\cdot\mathrm{ann}(d)$, there exists $z\in U_m(R)$ such that $r\cdot\mathrm{ann}(a)\cap r\cdot\mathrm{ann}(b)=r\cdot\mathrm{ann}(a+zb)$.*

(3) *Whenever $l\cdot\mathrm{ann}(a)\cap l\cdot\mathrm{ann}(b)=l\cdot\mathrm{ann}(d)$, there exists $y\in U_m(R)$ such that $l\cdot\mathrm{ann}(a)\cap l\cdot\mathrm{ann}(b)=l\cdot\mathrm{ann}(a+by)$.*

Proof. (1)⇒(2). Suppose $r\cdot\mathrm{ann}(a)\cap r\cdot\mathrm{ann}(b)=r\cdot\mathrm{ann}(d)$. By [5, Proposition 3.4], we claim that $Ra+Rb=Rd$. Using Proposition 12, we can find some $z\in U_m(R)$ such that $a+zb = du$ for some $u \in U(R)$. Therefore $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = r \cdot \text{ann}(d) = r \cdot \text{ann}(a+zb)$, as desired.

(2)⇒(1). Assume that $Ra+Rb = R$. Then $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = r \cdot \text{ann}(1)$. Thus, we claim that $r \cdot \text{ann}(a) \cap r \cdot \text{ann}(b) = r \cdot \text{ann}(a+zb)$ for a $z \in U_m(R)$. Therefore $r \cdot \text{ann}(1) = r \cdot \text{ann}(a+zb)$. By [5, Proposition 3.4], we show that $R = R(a+zb)$, and then $a+zb = u$ is left invertible in R. Assume that $vu = 1$ for some $v \in R$. From $Rv+R(1-uv) = R$, we also have $w \in U_m(R)$ such that $v+w(1-uv) = t$ is left invertible in R. Clearly, we have $tu = (v+w(1-uv))u = 1$. Hence t is a unit of R. Therefore $a+zb = u$ is a unit of R, as desired.

(1)⇔(2). By the symmetry of $(m,1)$-stable range condition, we complete the proof. □

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