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FREE OBJECTS IN THE CATEGORY OF GEOMETRIES

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ABSTRACT. The aim of this note is to introduce the class of free geometries purely in terms of morphisms. Several classes of well-known matroid morphisms are characterized via the new concept.

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1. Introduction. We shall assume familiarity with category and matroid theories; for an introduction, see [2, 3], respectively. In particular, a *matroid* M is an ordered pair $(E,\mathfrak{f}_M)$ where $\mathfrak{f}_M$ is a collection of subsets, called *flats* of M, of a finite set E such that E is a flat of M, the intersection of any two flats of M is a flat of M and if $F\in\mathfrak{f}_M$ and $\{F_1,F_2,\dots,F_k\}$ is the set of minimal members of $\mathfrak{f}_M$ (with respect to inclusion) that properly contain F (denoted by $F_i \succ F$), then $F_1\cup F_2\cup\cdots\cup F_k = E$. The set E is called the *ground set* of M. A formal notation for the matroid on the ground set E with flats $\mathfrak{f}_M$ is $M(E,\mathfrak{f}_M)$, but when no confusion will arise, we refer to this matroid as M. When several matroids $M(E_i,\mathfrak{f}_i)$, $i=1,2,\dots,n$ are being considered, we shall often denote these matroids by M_1, $M_2,\dots,M_n$.

By a *combinatorial geometry* we mean a loopless matroid with no multiple elements, that is, a matroid in which the empty set and each point, if any exists, is a flat. We will use the shorter “geometry” in place of “combinatorial geometry.” A *free geometry* is a geometry which has every subset of its ground set as a flat. If the ground set of a free geometry has m elements, then we denote that geometry by $U_{m,m}$.

Our main goal in this note is to introduce a categorical definition of a free object in the *category* $\mathcal{G}$ *of geometries and strong maps*. We define a functor, which we call a free functor, from the *subcategory* $\mathcal{F}$ *of free geometries* to the category $\mathcal{G}$. Lastly, we show that $\mathcal{F}$ is a coreflective subcategory of $\mathcal{G}$ and the free functor is a faithful functor which is a right adjoint of the inclusion functor.

The category of free geometries play an important role in solving the following open problem:

Find a finite set of elementary axioms that characterize the category $\mathcal{G}$.

It was that problem which prompted us to study the notion of free geometries.

2. Free objects. Define the *isthmus* 1 to be an object in $\mathcal{G}$ with exactly one endomorphism. Since in any category with an object M, the identity map i_M is always an endomorphism of M, i_1 is the endomorphism of 1. Observe that 1 is the terminal object of $\mathcal{G}$ while the initial object 0 is the empty geometry which is isomorphic to $U_{0,0}$.

Proposition 2.1. *For every object M, x is an element of M, or $x \in M$, if and only if x is a morphism with $1 \xrightarrow{x} M$.*

In the concrete category of geometries and strong maps, 1 is isomorphic to the free geometry $U_{1,1}$. Clearly $U_{1,1}$ has one element and one endomorphism. We shall denote the element of 1 by c. Next, we give a categorical definition of free objects which has never been introduced before. We remark that this definition is not obvious.

Definition 2.2. An object D is called a *free object* if for every $x \in D$ there exists a morphism h_x with $D \xrightarrow{h_x} 1 \amalg 1$ such that for every $y \in D$, $y \neq x$, we have $h_x x \neq h_x y$.

Clearly the objects 0, 1 are free. Also $1 \amalg 1$ is free since $1 \amalg 1$ has only two elements and the identity on $1 \amalg 1$ satisfies the property of h_x in the definition of free objects. We notice as 1 is isomorphic to $U_{1,1}$ that has a ground set isomorphic to $\{c\}$, the geometry $1 \amalg 1$ has a ground set isomorphic to $\{c_1, c_2\}$. Next, we state and prove our first main result.

Theorem 2.3. *A geometry is free if and only if it is isomorphic to $U_{n,n}$.*

Proof. Let $D = M(E, \mathfrak{F}_D)$ be a free geometry such that $|E| = n$. If $n = 0$, then $D \cong U_{0,0}$. If $n = 1$, then $D \cong 1 \cong U_{1,1}$. If $n \geq 2$, then to show $D \cong U_{n,n}$ it is sufficient to show that $\{x\}$ and $E \backslash \{x\}$ are flats of D for all $x \in E$. Since then, for every proper subset $F \subset E$ such that $|F| \leq n-1$, $F \in \mathfrak{F}_D$. Let $x \in E$. Then the constant map f_x with $1 \xrightarrow{f_x} D$ where $f_x(c) = x$ is a strong map. Hence as D is free, there exists a strong map h_x with $M \xrightarrow{h_x} 1 \amalg 1$ such that for every strong map g with $1 \xrightarrow{g} M$, $g \neq f_x$, we must have $h_x f_x \neq h_x g$. Thus assume $h_x f(c) = c_1$. Again as D is free and as for all $y \in E \backslash \{x\}$, the constant map f_y as above is a strong map such that $f_y \neq f_x$, we have $h_x f_x(c) \neq h_x f_y(c)$. Thus, $h_x(y) = c_2$ for all $y \in E \backslash \{x\}$. As $\{c_1\}$ and $\{c_2\}$ are flats of $1 \amalg 1$, $\{x\} = h_x^{-1}(\{c_1\})$ and $E \backslash \{x\} = h_x^{-1}(\{c_2\})$ are flats of D. Therefore, $D \cong U_{n,n}$.

If $D \cong U_{n,n}$ for some n where $U_{n,n}$ has a ground set $\acute{E}$, then for every strong map f with $1 \xrightarrow{f} D$, define a strong map h with $U_{n,n} \xrightarrow{h} 1 \amalg 1$ by $h(z) = c_1$ when $z = f(c)$, and $h(z) = c_2$ otherwise. For every strong map g with $1 \xrightarrow{g} M$ such that $g \neq f$, $g(c) \neq f(c)$ and hence $hg \neq hf$. Therefore, D is free. □

The proof of the following weak axiom of choice follows directly from the axiom of choice for sets.

Theorem 2.4. *For every morphism f with $M_1 \xrightarrow{f} D$ where $M_1 \ncong 0$ and D is a free object, there exists a morphism g with $D \xrightarrow{g} M_1$ such that $f = fgf$.*

Next, we show 1 is a generator and use that to give a sufficient condition for a morphism in the category $\mathcal{G}$ to be an epimorphism.

Lemma 2.5. *The isthmus object 1 is a generator.*

Proof. If M and N are objects and f, g are morphisms from M to N such that $f \neq g$, then there exists $x \in M$ (i.e., x is a morphism with $1 \xrightarrow{x} M$) such that $fx \neq gx$. Thus 1 is a generator. □

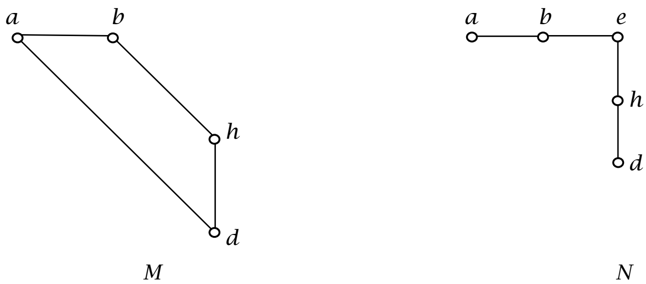

FIGURE 2.1. The converse of Proposition 2.5 does not hold.

PROPOSITION 2.6. *If M_1 and M_2 are two objects and f is a morphism with $M_1 \xrightarrow{f} M_2$ such that for every element $g \in M_2$ there exists an element $h \in M_1$ satisfying $g = fh$, then f is an epimorphism.*

PROOF. If N is an object and f_1, f_2 are two morphisms with $M_2 \xrightarrow{f_1} N$ and $M_2 \xrightarrow{f_2} N$ such that $f_1 f = f_2 f$, then we need only show the right cancellation law holds, that is, $f_1 = f_2$. Suppose $f_1 \neq f_2$. By Lemma 2.5, 1 is a generator and hence there exists a morphism m with $1 \xrightarrow{m} M_2$ such that $f_1 m \neq f_2 m$. Thus by assumption, there exists a morphism k with $1 \xrightarrow{k} M_1$ such that $m = fk$ and hence, $f_1 fk = f_1 m \neq f_2 m = f_2 fk$. That is, $f_1 fk \neq f_2 fk$ which is a contradiction to the fact that $f_1 fh = f_2 fh$ (because $f_1 f = f_2 f$). □

Next we show the converse of the preceding proposition need not hold in $\mathcal{G}$.

EXAMPLE 2.7. Consider the matroids M and N given by the point configurations in Figure 2.1. By [1, Proposition 3], the inclusion map i with $M \xrightarrow{i} N$ is an epimorphism. Define a strong map f with $1 \xrightarrow{f} N$ by $f(c) = e$. If g is a strong map with $1 \xrightarrow{g} M$ such that $f = ig$, then $g(c) = ig(c) = f(c) = e$ which is a contradiction to the fact that $e \notin E(M)$. Therefore, the converse of Proposition 2.6 does not hold.

3. Some peculiar morphisms.

In [1], Crapo proved that a strong map is a monomorphism if and only if it is a one to one map on points. It was also shown that an onto strong map, on points, is an epimorphism but an epimorphism need not be onto, on points. Next, we show that an epimorphism with free codomain is onto, on points.

PROPOSITION 3.1. *If f with $M \xrightarrow{f} D$ is an epimorphism where D is free, then f is an onto map on points.*

PROOF. Suppose $E(M)$ and $E(D)$ are the ground sets of M and D, respectively. If f is not onto, then there exists $x \in E(D)$ such that $x \notin f(E(M))$. Let H be a geometry on the set $\{x, y\}$ and define strong maps g and h from D to H by $g(x) = y$, $g(z) = x$ when $z \in f(E(M))$ and $h(z) = x$ for all $z \in E(D)$. Then $gf = hf$ and $g \neq h$. Therefore, f is not an epimorphism. □

PROPOSITION 3.2. *Every nonzero object M has elements and the morphism t with $M \xrightarrow{t} 1$ is an epimorphism.*

Proof. By Theorem 2.4, there exists a morphism h with $1 \xrightarrow{h} M$ such that $t = tht$ and hence $th = thth$. Therefore, $th = i_1$ and as i_1 is the only endomorphism of 1, by Proposition 2.6, t is an epimorphism. □

Proposition 3.3. *Any bimorphism (= a monomorphism and an epimorphism) f with $M_1 \xrightarrow{f} M_2$ with free domain and codomain is an isomorphism.*

Proof. If $M_1 \cong 0$, by Proposition 3.2, $M_2 \cong 0$ since f is an epimorphism. If $M_1 \not\cong 0$, by Theorem 2.4, there exists a morphism g with $M_2 \xrightarrow{g} M_1$ such that $f = fgf$ and since f is a bimorphism $gf = i_{M_1}$ and $fg = i_{M_2}$. Thus f is an isomorphism. □

The following theorem indicates that the category of free objects and strong maps is a coreflective subcategory of $\mathcal{G}$. The proof of that theorem is not hard and is thus left to the reader.

Theorem 3.4. *For every object M, there exists a free object $|M|$ together with a morphism t_M with $|M| \xrightarrow{t_M} M$ such that for every free object D and a morphism h with $D \xrightarrow{h} M$, there exists a unique morphism k with $D \xrightarrow{k} |M|$ such that $h = t_M k$. That is to say, the subcategory $\mathcal{F}$ of free geometries and strong maps is a coreflective subcategory of $\mathcal{G}$.*

Next, we state and prove several facts related to the morphisms t_M and the objects $|M|$, all purely in terms of morphisms only.

Proposition 3.5. *The morphism t_M is a bimorphism.*

Proof. By Theorem 3.4 and Proposition 2.6, t_M is an epimorphism. If x and y are elements of $|M|$ such that $t_M x = t_M y$, then as 1 is a free object, by Theorem 3.4, there exists a unique element $h \in |M|$ such that $t_M x = t_M h$. But as $t_M y = t_M x$, $x = h = y$ and hence t_M is a monomorphism. □

Observe that $|M|$ is defined up to isomorphism and the operation "$|\ |$" is a functor which we call the *free functor*. Next, we prove the free functor is a faithful functor that is also a right adjoint of the inclusion functor from $\mathcal{F}$ to $\mathcal{G}$.

Proposition 3.6. *The free functor is faithful, that is, for every morphisms f, g from M to N such that $|f| = |g|$, then $f = g$. Moreover, the free functor is a right adjoint of the inclusion functor $\mathcal{F} \hookrightarrow \mathcal{G}$.*

Proof. We prove the first part of the proposition and leave the other to the reader. If $|f| = |g|$, then $ft_M = t_N|f| = t_N|g| = gt_M$ and since t_M is an epimorphism, $f = g$. □

The proof of the following proposition is immediate and is left to the reader.

Proposition 3.7. *A morphism f is a bimorphism if and only if $|f|$ is a bimorphism.*

Corollary 3.8. *If f is a bimorphism with $D \xrightarrow{f} N$ where D is a free object, then $D \cong |N|$.*

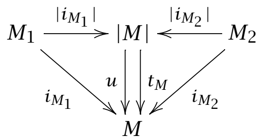

FIGURE 3.1. Coproduct of free objects is free.

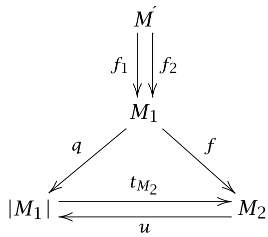

FIGURE 3.2. A regular epimorphism with free domain has a free codomain.

COROLLARY 3.9. *If f is a monomorphism with $M_1 \xrightarrow{f} M_2$ and M_2 is a free object, then M_1 is a free object.*

PROOF. For every morphism g with $1 \xrightarrow{g} M_1$, fg is a morphism from 1 to M_2 and as M_2 is a free object, there exists a morphism h_{fg} from M_2 to $1 \amalg 1$ such that $h_{fg}fg \neq h_{fg}m$ for every morphism m with $1 \xrightarrow{m} M_2$ such that $m \neq fg$. Let e be a morphism with $1 \xrightarrow{e} M_1$ such that $e \neq g$. Then as f is a monomorphism, $fe \neq fg$. Thus $h_{fg}fe \neq h_{fg}fg$. Therefore, $hf \neq hg$ where $h = h_{fg}f$ and hence M_1 is a free object. □

COROLLARY 3.10. *If M_1 and M_2 are free objects, then $M_1 \amalg M_2$ is a free object.*

PROOF. As M_1 and M_2 are free objects, $M_1 \cong |M_1|$ and $M_2 \cong |M_2|$. By definition of the coproduct $M_1 \amalg M_2$, there exists a unique morphism u with $M \xrightarrow{u} |M|$, where $M \cong M_1 \amalg M_2$, such that $|i_{M_1}| = ui_{M_1}$ and $|i_{M_2}| = ui_{M_2}$. (See the diagram in Figure 3.1.) Thus $t_M ui_{M_1} = t_M|i_{M_1}|$ and $t_M ui_{M_2} = t_M|i_{M_2}|$. But by definition of $|i_{M_1}|$ and $|i_{M_2}|$, we have $t_M|i_{M_1}| = i_{M_1}$ and $t_M|i_{M_2}| = i_{M_2}$. Therefore, $i_M = t_M u$ and then as i_M is a monomorphism, u is a monomorphism. Thus since $|M|$ is a free object, by Corollary 3.9, M is a free object. □

COROLLARY 3.11. *A regular epimorphism with a free domain has a free codomain.*

PROOF. If f is a morphism with $M_1 \xrightarrow{f} M_2$ where M_1 is a free object, M' is an object and f_1, f_2 are two morphisms from M' to M_1 such that $\langle M_1, f\rangle$ is isomorphic to the coequalizer $\mathrm{Coeq}(f_1,f_2)$ of f_1 and f_2, then by Theorem 3.4, there exists a unique morphism q with $M_1 \xrightarrow{q} |M_2|$ such that $f = t_{M_2}q$. (See the diagram in Figure 3.2.) Thus, $t_{M_2}qf_1 = ff_1 = ff_2 = t_{M_2}qf_2$ and since t_{M_2} is a monomorphism, $qf_1 = qf_2$. By definition of the coequalizer $\mathrm{Coeq}(f_1,f_2)$, there exists a unique morphism u with $M_2 \xrightarrow{u} |M_2|$ such that $q = uf$. Thus $i_{M_2}f = f = t_{M_2}q = t_{M_2}uf$. Since f is an epimorphism, $i_{M_2} = t_{M_2}u$ and as i_{M_2} is a monomorphism, u is a monomorphism. As $|M_2|$ is a free object, by Corollary 3.9, M_2 is a free object. □

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