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# BOUNDARY VALUE PROBLEM FOR $r^2 d^2f/dr^2 + f = f^3$ (I): EXISTENCE AND UNIQUENESS

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Abstract. We study the equation $r^2 d^2f/dr^2 + f = f^3$ with the boundary conditions $f(1) = 0$, $f(\infty) = 1$, and $f(r) > 0$ for $1 < r < \infty$. The existence of the solution is proved using a topological shooting argument. And the uniqueness is proved by a variation method.

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**1. Introduction.** Consider the following boundary value problem

$$r^2 f'' + f = f^3, \quad 0 < r < \infty, \tag{1.1}$$

$$f(r) \longrightarrow 0, \quad \text{as } r \longrightarrow 0, \tag{1.2}$$

$$f(\infty) = 1, \tag{1.3}$$

where $'$ means $d/dr$. This problem was proposed in [11] for studying the monopole solution in the pure SU(2) gauge field theory. The solution to this problem is usually called Wu-Yang solution. And when people study the Yang-Mills coupled equations, for example, in [2, 4, 8], this equation is always considered. The regular solution to this equation only comes out from this boundary value problem or equivalently $f(\infty) = -1$ (see [2]). So it is useful to give a complete study for the existence, uniqueness, asymptotics and connection formulas for the parameters in the asymptotic formulas. The readers who are interested in the physics background are referred to [1, 5].

Wu and Yang [11], Protogenov [7], and Breitenlohner et al. [2] obtained that the solution to this boundary value problem has the asymptotics

$$f(r) \sim \alpha r^{1/2} \sin\left(\frac{\sqrt{3}}{2}\log r + \beta\right), \tag{1.4}$$

as $r \to 0$, and

$$f(r) \sim 1 + \frac{\gamma}{r}, \tag{1.5}$$

as $r \to \infty$ for some parameters $\alpha, \beta, \gamma$. The current work is motivated to find the formulas for the parameters $\alpha, \beta$, and $\gamma$, which are called connection formulas for this problem.

In this paper and in [9, 10], we study this boundary value problem and finally give the connection formulas. We will show that any solution to this problem has infinitely many zeros, and the zeros have upper bound. So the largest zero $r = r_0$ exists. Since the equation is invariant under the scaling transformation $r \to cr$, we just need to discuss $r_0 = 1$. To study the boundary value problem, we first consider the existence and uniqueness of another boundary value problem $f(1) = 0$, $f(\infty) = 1$, and $f(r) > 0$ for $r > 1$, which is the work of this paper. Using shooting arguments and variation methods, we prove that this problem has a unique solution, and the solution has asymptotics

$$f(r) \sim a^* \log r, \tag{1.6}$$

as $r \to 1$, for a positive constant $a^*$.

In [9] we will find exact formula for this number $a^*$ using an analytic continuity method to study the analytic property (in complex domain) of the solution at $r = \infty$ and extend the property to $r = 1$. In [10], we will discuss the global solution to the boundary value problem (1.1), (1.2), and (1.3), and give the asymptotics and connection formulas. The method used in these papers would be applicable to study other equations, for example, $r^2 f'' = F(f)$, where $F(f)$ is a polynomial of $f$.

To study (1.1) we put

$$r = e^x, \qquad f(r) = y(x). \tag{1.7}$$

Then (1.1) is changed to

$$y'' - y' + y = y^3. \tag{1.8}$$

In Section 2, we prove that there is a solution $y^*(x)$ to (1.8) for $x > 0$, such that $y^*(0) = 0$, $y^*(\infty) = 1$, and $y^*(x) > 0$, $x > 0$. The method we use in this paper is the one-dimensional shooting argument which has been widely used to discuss boundary value problems. In Section 3, we show that the solution $y^*(x)$ $(x > 0)$ is unique and is strictly monotone by using variation methods. In the next paper [9], we will discuss the number $a^* = y^{*\prime}(0)$, which will be used to analyze the global solution $(-\infty < x < \infty)$ [10].

## 2. Existence of the solution.

We consider the following problem

$$y'' - y' = y^3 - y, \quad 0 < x < \infty, \qquad y(0) = 0, \qquad y'(0) = a. \tag{2.1}$$

In this section, we show that there is a positive value of $a$, such that the solution to (2.1) satisfies

$$y(\infty) = 1, \qquad y'(x) > 0, \quad 0 < x < \infty. \tag{2.2}$$

First we state a basic fact in ordinary differential equations [3].

**Lemma 2.1.** *For any $a$, there is a unique bounded solution $y(x,a)$ to (2.1) in a neighborhood of 0. In particular, when $a = 0$, $y \equiv 0$.*

We then analyze the behaviour of the solution when $a$ is large or small. It will be shown below that when $a$ is large, $y$ crosses 1 before $y'$ crosses 0, and when $a$ is small, $y'$ crosses 0 before $y$ crosses 1. Then we show that there is a value of $a$, such that $y(x,a)$ does not cross 1, and $y'(x,a)$ does not cross 0, and $y(\infty,a) = 1$. This is the so called shooting method.

**Lemma 2.2.** *When $a > 1/\sqrt{2}$, the solution $y(x)$ to (2.1) satisfies*

$$y(x^+) > 1, \quad y'(x) > 0, \quad 0 \le x \le x^+, \tag{2.3}$$

*where $x^+ = (a^2 - 1/2)^{-1/2} + 1$.*

**Proof.** Let

$$v(x) = 1 - y(x). \tag{2.4}$$

Then (2.1) becomes

$$v'' - v' = 2v - 3v^2 + v^3, \tag{2.5}$$

$$v(0) = 1, \qquad v'(0) = -a. \tag{2.6}$$

Multiplying (2.5) by $v'$ and integrating, we obtain

$$v'^2 = 2v^2\left(1 - \frac{1}{2}v\right)^2 + \left(a^2 - \frac{1}{2}\right) + 2\int_0^x \left(v'(s)\right)^2 ds, \tag{2.7}$$

for $x > 0$. When $a > 1/\sqrt{2}$, the right-hand side of (2.7) is always positive. And because $v'(0) = -a < 0$, we have

$$v'(x) = -\left(2v^2\left(1 - \frac{1}{2}v\right)^2 + \left(a^2 - \frac{1}{2}\right) + \int_0^x v'(s)^2 ds\right)^{1/2} < -\left(a^2 - \frac{1}{2}\right)^{1/2}. \tag{2.8}$$

Hence

$$\begin{aligned} v(x^+) &= v(0) + \int_0^{x^+} v'(s)\,ds < 1 - \int_0^{x^+} \sqrt{a^2 - \frac{1}{2}}\,ds \\ &= -\sqrt{a^2 - \frac{1}{2}} < 0, \quad v'(x) < 0,\ 0 \le x \le x^+, \end{aligned} \tag{2.9}$$

or equivalently

$$y'(x^+) > 1, \quad y'(x) > 0, \quad 0 \le x \le x^+. \tag{2.10}$$

So the lemma is proved. □

**Lemma 2.3.** *There is $a^- > 0$, such that if $a \in (0, a^-]$, the solution $y(x) = y(x,a)$ satisfies*

$$y'(x^-) < 0, \quad y(x) > 0, \quad 0 < x \le x^-, \tag{2.11}$$

*where $x^- = 5\pi/3\sqrt{3}$.*

**Proof.** Let

$$y(x) = aw(x). \tag{2.12}$$

Then (2.1) becomes

$$w'' - w' + w = a^2 w^3, \qquad w(0) = 0, \qquad w'(0) = 1. \tag{2.13}$$

As $a \to 0$, $w(x)$ uniformly tends, on compact intervals in $x$, to the solution of the problem

$$W'' - W' + W = 0, \qquad W(0) = 0, \qquad W'(0) = 1. \tag{2.14}$$

It is not hard to see that the solution of this problem is

$$W(x) = \frac{2}{\sqrt{3}} e^{(1/2)x} \sin\left(\frac{\sqrt{3}}{2}x\right). \tag{2.15}$$

We see that

$$W'(x^-) < 0, \quad W(x) > 0, \quad 0 < x \le x^-. \tag{2.16}$$

Thus there exists $a^- > 0$, such that if $a \in (0, a^-]$, there is

$$w'(x^-) < 0, \quad w(x) > 0, \quad 0 < x \le x^-. \tag{2.17}$$

By (2.12), the lemma is proved. □

For the solution $y(x,a)$ of (2.1), where again $a = y'(0)$, we define

$$\begin{aligned} S^+ &= \{a > 0 \mid y \text{ crosses 1 before } y' \text{ crosses } 0\}, \\ S^- &= \{a > 0 \mid y' \text{ crosses 0 before } y \text{ crosses } 1\}, \end{aligned} \tag{2.18}$$

Lemma 2.2 shows that $(1/\sqrt{2}, \infty) \subset S^+$, and Lemma 2.3 shows that $(0, a^-) \subset S^-$.

**Theorem 2.4.** *There is a solution to the following problem*

$$y'' - y' = y^3 - y, \quad 0 < x < \infty, \tag{2.19a}$$

$$y(0) = 0, \qquad y(\infty) = 1, \tag{2.19b}$$

$$y(x) > 0, \quad 0 < x < \infty. \tag{2.19c}$$

**Proof.** By Lemmas 2.2 and 2.3, we have that $S^+$ and $S^-$ are nonempty sets. By the definition of $S^-$ and $S^+$, we see that they are disjoint sets. By the implicit function theorem, it is not difficult to show that $S^+$ and $S^-$ are open sets. Thus

$$(0, \infty) \setminus (S^- \cup S^+) \ne \emptyset. \tag{2.20}$$

Hence there is $a^* > 0$, $a^* \notin S^- \cup S^+$, such that $y(x, a^*)$ satisfies

$$y'(x, a^*) > 0, \quad y(x, a^*) < 1, \quad 0 < x < \infty. \tag{2.21}$$

So $y(\infty, a^*) = b$, where $0 < b \le 1$.

We show $b = 1$. There exists $x_0 > 0$ such that when $x_0 < x < \infty$,

$$\frac{b}{2} < y(x) < b. \tag{2.22}$$

If $b < 1$, we have from (2.19a)

$$\begin{aligned} y'(x) &= e^x \int_{x_0}^x e^{-s} y(s)\left(y^2(s) - 1\right) ds \\ &\le e^x \int_{x_0}^x e^{-s} \frac{b}{2}\left(b^2 - 1\right) ds = \frac{b(b^2-1)}{2}\left(e^{x - x_0} - 1\right) \longrightarrow -\infty, \end{aligned} \tag{2.23}$$

as $x \to \infty$, which is a contradiction, since $y(\infty, a^*)$ exists. So $b = 1$. □

**3. Uniqueness of the solution.** In Section 2, we have proved that problem (2.19) has a solution. Now, we show that the solution is also unique. Start with the following lemma.

**Lemma 3.1.** *If $y(x)$ is a solution to the problem (2.19), that is,*

$$y'' - y' + y = y^3, \quad 0 < x < \infty, \tag{3.1}$$

$$y(0) = 0, \qquad y(\infty) = 1, \tag{3.2}$$

$$y(x) > 0, \quad 0 < x < \infty, \tag{3.3}$$

*then $y(x)$ has the following properties*

(i) $y(x) < 1$, $0 \le x < \infty$.

(ii) $y'(x) > 0$, $0 \le x < \infty$. *And* $y'(\infty) = 0$.

(iii) $y(x) = 1 - ce^{-x} + O(e^{-2x})$, $y'(x) = ce^{-x} + O(e^{-2x})$, *and then* $y'(x)/(y(x)-1) = -1 + O(e^{-x})$, *as* $x \to \infty$, *where* $c > 0$.

**Proof.** (i) Suppose $x_1 > 0$ is the first point, such that $y(x_1) = 1$. By the uniqueness of the solution, $y'(x_1) > 0$. We then claim that $y'(x) > 0$, for $x_1 < x < \infty$. If not, suppose $x_2 > x_1$ is the first point such that $y'(x_2) = 0$. Then since $y'(x) > 0$, for $x_1 < x < x_2$, there is

$$y'' = y' + y(y^2 - 1) > 0, \qquad y'(x_2) = y'(x_1) + \int_{x_1}^{x_2} y''(s)\,ds > y'(x_1) > 0, \tag{3.4}$$

which is a contradiction. So $y'(x) > 0$, for $x_1 < x < \infty$. Then we cannot get $y(\infty) = 1$, which is a contradiction. Thus $y < 1$.

(ii) If $y'(0) = 0$, then since $y(0) = 0$, by the uniqueness of the solution, we see that $y \equiv 0$, which is a contradiction. So $y'(0) > 0$.

Now suppose $x_3 > 0$ is the first point, such that $y'(x_3) = 0$. By (3.1) and (i), there is

$$y'(x) = e^x \int_{x_3}^{x} e^{-s} y(s)\left(y^2(s) - 1\right) ds < 0, \tag{3.5}$$

for $x > x_3$. Then since $y(x_3) < 1$, we cannot have $y(\infty) = 1$. This is a contradiction. So $y' > 0$, for all $x \ge 0$.

If we do not have $y'(\infty) = 0$, then there are (small) $\epsilon > 0$, (large) $x_0 > 0$, such that $y'(x_0) > \epsilon$, and $y(x) - y^3(x) < \epsilon$ for $x \ge x_0$. By (3.1) we have

$$y''(x_0) = y'(x_0) - \left(y(x_0) - y^3(x_0)\right) > 0, \tag{3.6}$$

which implies that $y''(x) > 0$ in a neighborhood of $x_0$. So $y'(x)$ is increasing in this neighborhood. By (3.1)

$$y''(x) = y'(x) - \left(y(x) - y^3(x)\right) > y'(x) - \epsilon \ge y'(x_0) - \epsilon > 0, \tag{3.7}$$

when $x \ge x_0$. We see that $y''(x)$ remains positive, and $y'(x)$ keeps increasing for $x \ge x_0$, which is a contradiction since $y(\infty) = 1$. Therefore we have $y'(\infty) = 0$.

(iii) Let $y_1 = y$, $y_2 = y'$, and change (3.1) into the system

$$y_1' = y_2, \qquad y_2' = y_2 - y_1 + y_1^3. \tag{3.8}$$

It is easy to see that $(1,0)$ is a saddle point in the phase plane. Since $(y_1(\infty), y_2(\infty)) = (1,0)$, by the stable manifold theorem (see [3, 6]) we get that as $x \to \infty$, $(y_1(x), y_2(x))$ lies on the stable manifold. And by a standard argument [6] we have

$$y(x) = 1 - ce^{-x} + O(e^{-2x}), \qquad y'(x) = de^{-x} + O(e^{-2x}), \tag{3.9}$$

as $x \to \infty$, for some constants $c, d$. Because $y(x) < 1$ for $x > 0$, $c$ cannot be negative. If $c = 0$, we convert (3.1) into an integral equation by Green's function. By a contraction argument we get $y \equiv 1$, which is a contradiction. Thus $c > 0$. By

$$y(x) = 1 - \int_x^\infty y'(s)\,ds, \tag{3.10}$$

we see that $c = d$. □

To prove the uniqueness, we use a variational method. Suppose $y(x, a_1)$ is a solution to (2.19). We show in this section that when $a$ increases a little from $a_1$, $y(x,a)$ crosses 1 at some point. We then show the root $x$ of $y(x,a) = 1$ is moving left while $a$ is increasing further, which means that for any $a > a_1$, $y(x,a)$ does not satisfy $y(\infty, a) = 1$, that is, they are not solutions to (2.19). So we can show the solution is unique.

For simplicity we do not directly discuss (2.19). Instead we consider the original equation (1.1). Suppose $f(r,a)$ is a solution to the following problem

$$r^2 f'' + f = f^3, \quad 1 < r < \infty, \tag{3.11}$$

$$f(1) = 0, \qquad f'(1) = a, \tag{3.12}$$

where $a > 0$. It is easy to see that (3.11) and (3.12) are equivalent to (2.1). Define

$$\psi(r,a) = \frac{\partial f(r,a)}{\partial a}. \tag{3.13}$$

**Lemma 3.2.** (i) *If $f(r,a)$ crosses 1 at a point $r = r_1 > 1$, and $f(r,a) > 0$ for $1 < r \le r_1$, then there is*

$$\psi(r_1, a) > 0, \tag{3.14}$$

*for $1 < r \le r_1$.*

(ii) *If for some $a = a_1$, $f(r, a_1)$ satisfies $f(r,a_1) > 0$, for $r > 1$, and $f(\infty, a_1) = 1$, then*

$$\psi(r, a_1) > 0, \qquad \psi'(r, a_1) > 0, \tag{3.15}$$

*for $r > 1$.*

**Proof.** (i) By the definition of $\psi$ (3.13), $\psi$ satisfies

$$r^2 \psi'' + \psi = 3f^2 \psi, \tag{3.16}$$

$$\psi(1) = 0, \qquad \psi'(1) = 1. \tag{3.17}$$

By (3.11) and (3.16), we have

$$r^2(f'\psi - f\psi')' = -2f^3\psi. \tag{3.18}$$

Assume for contradiction $r_0 \in (1,r_1]$ is the first point, such that $\psi(r_0,a) = 0$. Then since $r_0$ is the first zero of $\psi(r,a)$ after $r = 1$, and $\psi(1,a) = 1 > 0$, we have $\psi'(r_0,a) \le 0$. If $\psi'(r_0,a) = 0$, by the uniqueness of solution, $\psi(r,a) \equiv 0$, which is a contradiction. So $\psi'(r_0,a) < 0$. Then since $f(1) = \psi(1) = 0$, we get from (3.18)

$$f'(r_0)\psi(r_0) - f(r_0)\psi'(r_0) = -2\int_1^{r_0} \frac{f^3(s)\psi(s)}{s^2}\,ds < 0. \tag{3.19}$$

By the assumption $\psi(r_0) = 0$, we obtain

$$-f(r_0)\psi'(r_0) < 0. \tag{3.20}$$

Then $f(r_0) > 0$ implies that $\psi'(r_0) > 0$, which is a contradiction. So (3.14) is true.

(ii) Recalling the relation between $y(x,a)$ and $f(r,a)$ (1.7), and by Lemma 3.1(i), we have $0 < f(r,a_1) < 1$ for $r > 1$, which implies $f(r,a_1)$ has no singularity in $(1,\infty)$. Then by the same argument as above, we have $\psi(r,a_1) > 0$ for $r > 1$. Now suppose $r_2 > 1$ is the first point such that $\psi'(r_2,a_1) = 0$, then we have

$$f'(r_2)\psi(r_2) = f(r_2)\psi'(r_2) - 2\int_1^{r_2} \frac{f^3(s)\psi(s)}{s^2}\,ds < 0. \tag{3.21}$$

This is a contradiction because $\psi(r_2) > 0$, and $f'(r_2) > 0$ by Lemma 3.1(ii). So the lemma is proved. □

**Lemma 3.3.** *If $f(r,a_1)$ is a solution to (3.11) and (3.12), satisfying $f(\infty,a_1) = 1$ and $f(r,a_1) > 0$ for $r > 1$, then there exists $\bar{\epsilon} > 0$, such that for any $\epsilon \in (0,\bar{\epsilon}]$, $f(r,a_1+\epsilon)$ crosses 1 at some point $r_0 > 1$, and $f(r,a_1+\epsilon) > 0$ for $1 < r \le r_0$.*

**Proof.** By Lemmas 3.2(ii) and 3.1(ii), there exist $\bar{\epsilon} > 0$, $r_2 > r_1 > 1$, such that

$$f(r,a) > f(r,a_1), \qquad f'(r,a) > f'(r,a_1), \tag{3.22}$$

for $1 < r \le r_2$, $a_1 < a \le a_1 + \bar{\epsilon}$, and

$$f(r,a_1) > \frac{1}{\sqrt{2}}, \tag{3.23}$$

for $r \ge r_1$. Let $v(r,a) = f(r,a) - f(r,a_1)$ for $a_1 < a \le a_1 + \bar{\epsilon}$. Then $v$ satisfies the equation

$$r^2 v'' = (f^2 + ff_1 + f_1^2 - 1)v, \tag{3.24}$$

where $r > 1$. When $r_1 \le r \le r_2$, $a_1 < a \le a_1 + \bar{\epsilon}$, by (3.22), (3.23), and (3.24) we have

$$v(r,a) > 0, \qquad v'(r,a) > 0, \qquad v''(r,a) > 0. \tag{3.25}$$

By (3.23) and (3.24), we see that (3.25) is true for all $r \geq r_1$. Therefore we have

$$f(r,a) > f(r,a_1) + (f(r_1,a) - f(r_1,a_1)), \tag{3.26}$$

for all $r \geq r_1$. Since $f(\infty, a_1) = 1$, $f(r_1,a) - f(r_1,a_1) > 0$, we then conclude that $f(r,a)$ crosses 1 at some point for all $a \in (a_1, a_1 + \bar{\epsilon}]$, and $f(r,a)$ remains positive and finite before it crosses 1. □

**Theorem 3.4.** *Suppose that $f(r,a)$ is a solution to (3.11) and (3.12). There is a unique value $a = a^*$, such that $f(\infty, a^*) = 1$, and $f(r,a) > 0$ for $r > 1$. Thus the problem (2.19) has a unique solution $y^*(x) = y(x,a^*)$, and $y^*(x)$ has the asymptotics*

$$y^*(x) \sim a^* x, \tag{3.27}$$

*as $x \to 0$, and*

$$y^*(x) = 1 - ce^{-x} + O(e^{-2x}), \tag{3.28}$$

*as $x \to +\infty$.*

**Proof.** Theorem 2.4 has shown that such value of $a$ exists. Now suppose $a_1 > 0$ is a value of $a$, such that $y(x,a_1)$ solves (2.19). We want to show for any $a > a_1$, $y(x,a)$ does not satisfy (2.19).

Consider (3.11). Set

$$D = \{a > a_1 \mid f(r,a) = 1 \text{ for some } r > 1\}. \tag{3.29}$$

By Lemma 3.3, $(a_1, a_1 + \bar{\epsilon}] \subset D$. Let $r_1 = r_1(a) > 1$ be the least root of $f(r,a) = 1$, for $a \in D$. By the implicit function theorem, $r_1$ is a differentiable function of $a$ on $D$ and

$$f'(r_1,a)\frac{dr_1(a)}{da} + \psi(r_1,a) = 0. \tag{3.30}$$

By Lemma 3.3, the conditions in Lemma 3.2(i) are satisfied for $a = a_1 + \bar{\epsilon}$. So $\psi(r_1, a_1 + \bar{\epsilon}) > 0$. Since $f'(r_1, a_1 + \bar{\epsilon}) > 0$ ($f(r, a_1 + \bar{\epsilon})$ crosses 1 at $r_1$) it follows that

$$\frac{dr_1(a)}{da} < 0, \tag{3.31}$$

for $a = a_1 + \bar{\epsilon}$. Lemma 3.2(i) implies (3.31) is true for all $a > a_1 + \bar{\epsilon}$. Therefore as $a$ increases $r_1(a)$ monotonically decreases. Thus $D = (a_1, \infty)$, which means for $a > a_1$, $y(x,a)$ does not satisfy (2.19) by Lemma 3.1. If there is another value of $a$ which is less than $a_1$, such that $y(x,a)$ satisfies (2.19), then by the above argument, $y(x,a_1)$ does not satisfy (2.19), which is a contradiction. So we have proved the theorem. □

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## References

[1] A. Actor, *Classical solutions of* SU(2) *Yang-Mills theories*, Rev. Modern Phys. **51** (1979), no. 3, 461–525. MR 81b:81014.

[2] P. Breitenlohner, P. Forgács, and D. Maison, *Static spherically symmetric solutions of the Einstein-Yang-Mills equations*, Comm. Math. Phys. **163** (1994), no. 1, 141–172. MR 95b:83021. Zbl 809.53081.

[3] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955. MR 16,1022b. Zbl 064.33002.

[4] S. P. Hastings, J. B. McLeod, and W. C. Troy, *Static spherically symmetric solutions of a Yang-Mills field coupled to a dilaton*, Proc. Roy. Soc. London Ser. A **449** (1995), no. 1937, 479–491. MR 96b:53045. Zbl 840.34016.

[5] E. Malec, *On classical solutions of nonabelian gauge theories*, Acta Phys. Polon. B **18** (1987), no. 11, 1017–1064. MR 89d:81063.

[6] L. Perko, *Differential Equations and Dynamical Systems*, 2nd ed., Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 1996. MR 97g:34002. Zbl 854.34001.

[7] A. P. Protogenov, *Bag and multimeron solution of the classical Yang-Mills equation*, Phys. Lett. B **87** (1979), 80–82.

[8] J. A. Smoller, A. G. Wasserman, S.-T. Yau, and J. B. McLeod, *Smooth static solutions of the Einstein/Yang-Mills equations*, Comm. Math. Phys. **143** (1991), no. 1, 115–147. MR 93a:58044. Zbl 755.53061.

[9] C. B. Wang, *Boundary value problem for* $r^2 d^2 f/dr^2 + f = f^3$ *(II): connection formula*, http://xxx.lanl.gov/abs/math-ph/9903023.

[10] ———, *Boundary Value Problem for* $r^2 d^2 f/dr^2 + f = f^3$ *(III): global solution and asymptotics*, http://xxx.lanl.gov/abs/math-ph/9903024.

[11] T. T. Wu and C. N. Yang, *Some solutions of the classical isotopic gauge field equations*, Properties of Matter Under Unusual Conditions (H. Mark and S. Fernbach, eds.), Interscience, New York, 1969, pp. 349–354.

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