IJMMS 26:10 (2001) 635–641
PII. S0161171201005567
http://ijmms.hindawi.com

SOME RESULTS ON COINCIDENCE AND FIXED POINT THEOREMS FOR GENERALIZED CONTRACTION TYPE MAPPINGS

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(Received 24 April 1996 and in revised form 1 October 1996)

ABSTRACT. Some coincidence and fixed point theorems are proved for certain generalized contraction type single-valued and set-valued compatible mappings.

2000 Mathematics Subject Classification. 47H10, 54H25.

1. Introduction. Jungck [1] generalized the Banach contraction principle using the commuting map concept, which is extended by Sessa [4] giving weakly commuting map concept; this again modified in [2] by compatibility condition. Several authors [3, 5, 6] discussed various results on coincidence and fixed point theorem for compatible single-valued and multi-valued maps. Here we develop some coincidence and fixed point theorems for compatible single-valued and multi-valued maps satisfying some generalized contraction type condition. Henceforth, we denote by $\mathbb{N}$ and $\mathbb{R}_+$, the set of naturals and nonnegative reals, respectively, and $\omega = \mathbb{N} \cup \{0\}$ and (X,d), a metric space, unless otherwise stated.

2. Preliminaries

Definition 2.1 (see [3]). Two mappings $f,g : X \to X$ are compatible if and only if $d(fgx_n, gfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $fx_n \to t$, $gx_n \to t$, $t \in X$.

Let $C(X)$ = class of closed subsets of X, $\mathrm{CB}(X)$ = class of closed bounded subsets of X, $\mathrm{co}(K)$ = convex hull of $K \subset X$. The Hausdorff metric H on $\mathrm{CB}(X)$ is defined as $H(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{X\in B} D(x,A)\}$, for all $A,B \in \mathrm{CB}(X)$, where $D(x,A) = \inf_{y\in A} d(x,y)$.

Definition 2.2 (see [3]). The maps $f : X \to X$ and $T : X \to \mathrm{CB}(X)$ are compatible if and only if $fTx \in \mathrm{CB}(X)$ for all $x \in X$ and $H(fTx_n, Tfx_n) \to 0$ whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to M \in \mathrm{CB}(X)$, $fx_n \to t \in M$, where H is the Hausdorff metric on X.

We now recall the following lemmas.

Lemma 2.3 (see [7]). *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing upper semi-continuous (u.s.c.) function. Then $h(t) < t$ if and only if $h^n(t) \to 0$ for each $t > 0$ where h^n denotes the composition of h with itself n times.*

Lemma 2.4 (see [3]). *Let $T : X \to \mathrm{CB}(X)$ and $f : X \to X$ be compatible. If $fz \in Tz$ for some $z \in X$, then $fTz = Tfz$.*

3. Coincidence and fixed point theorems for single-valued maps

Theorem 3.1. *Let X be any nonempty set and (Y,d) be a complete metric space. Let $f,g,T : X \to Y$ satisfy*

(i) $f(X), g(X) \subseteq T(X)$;

(ii) *$T(X)$ is closed in Y;*

(iii) *for all $x,y \in X$,*

$$d(fx,gy) \le \varphi[\max\{d(Tx,Ty),d(Tx,fx),d(Tx,gy),d(Ty,fx),d(Ty,gy)\}], \tag{3.1}$$

where $h(t) = \varphi[\max\{t,t,at,bt,t\}] < t$, for each $t > 0$, $a,b \in \{0,1,2\}$ with $a+b = 2$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing u.s.c function. Then f,g,T have a coincidence point in X.

Further if

(iv) *f or g is injective, then the coincidence point is unique in X.*

Proof. Choose any $x_0 \in X$. From (i), we define an iteration $y_{2n} = fx_{2n} = Tx_{2n+1}$, $y_{2n+1} = gx_{2n+1} = Tx_{2n+2}$. Let $d_n = d(Tx_n, Tx_{n+1})$. Then from (iii), we have

$$\begin{aligned} d_{2n+1} &= d(Tx_{2n+1},Tx_{2n+2}) = d(y_{2n},y_{2n+1}) = d(fx_{2n},gx_{2n+1}) \\ &\le \varphi\Big[\max\Big\{d(Tx_{2n},Tx_{2n+1}),d(Tx_{2n},fx_{2n}), \\ &\qquad d(Tx_{2n},gx_{2n+1}),d(Tx_{2n+1},fx_{2n}),d(Tx_{2n+1},gx_{2n+1})\Big\}\Big] \\ &\le \varphi\Big[\max\Big\{d_{2n},d_{2n},(d_{2n}+d_{2n+1}),0,d_{2n}\Big\}\Big]. \end{aligned} \tag{3.2}$$

If $d_{2n+1} > d_{2n}$ then contradiction arises; so taking $d_{2n+1} \le d_{2n}$, we have $d_{2n+1} \le h(d_{2n})$. Similarly, $d_{2n+2} \le d_{2n+1}, d_{2n+2} \le h(d_{2n+1})$. Hence $d_{n+1} \le d_n$ and $d_n \le h(d_{n-1}) \le \cdots \le h^n(d_0)$, for all $n \in \omega$.

This yields, by Lemma 2.3, $\lim_n d_n = 0 = \lim_n d(y_n,y_{n+1})$. Now, the sequence $\{y_n\}$ is a Cauchy sequence in $f(X)$, which can be proved using the same technique as used in [6, Theorem 2.1] so from (ii), $\exists u \in X \ni \lim_n y_n = Tu$, that is, $\lim_n Tx_n = Tu$ and $\lim_n fx_{2n} = Tu = \lim_n gx_{2n+1}$. Suppose that $fu \ne Tu \ne gu$. Then

$$\begin{aligned} d(fu,Tu) &\le d(fu,gx_{2n+1}) + d(gx_{2n+1},Tu) \\ &\le \varphi\Big[\max\Big\{d(Tu,Tx_{2n+1}),d(Tu,fu),d(Tu,gx_{2n+1}),d(Tx_{2n+1},fu), \\ &\qquad d(Tx_{2n+1},gx_{2n+1})\Big\}\Big] + d(gx_{2n+1},Tu) \Longrightarrow d(fu,Tu) \\ &\le \varphi\Big[\max\Big\{0,d(Tu,fu),0,d(Tu,fu),0\Big\}\Big], \end{aligned} \tag{3.3}$$

as $n \to \infty$; hence $d(fu,Tu) < d(fu,Tu)$ which is absurd. Hence $fu = Tu$. Similarly, $gu = Tu$. Thus, $fu = Tu = gu$ and uniqueness of u follows from (iii) and (iv). □

Lemma 3.2. *Let $f,g:X\to X$ be compatible. If $fz=gz$ for some $z\in X$, then $fgz=gfz$.*

Proof. The proof is similar to that of Kaneko and Sessa [3]. □

Theorem 3.3. *Let (X,d,δ) be a bimetric space such that X is complete with respect to δ. Let $f,g,T:X\to X$ satisfy conditions (i)–(iii) of Theorem 3.1 with respect to d, and*

(v) *(f,T) and (g,T) are compatible pairs;*

(vi) *$\delta(x,y)\le k(d(x,y))$ for all $x,y\in X$,*

where $k:\mathbb{R}_+\to\mathbb{R}_+$ is continuous with $k(0)=0$. Then f,g,T have a unique common fixed point in X.

Proof. By Theorem 3.1, $\{Tx_n\}$ is Cauchy with respect to d and hence from (vi) it is Cauchy with respect to δ. Since X is complete with respect to δ, from Theorem 3.1(ii), there exists $z\in X\ni fz=Tz=gz$. Thus, by Lemma 3.2 and (v), $Tfz=fTz$ and $gTz=Tgz$. So $TTz=Tfz=fTz=ffz=fgz=gfz=ggz=gTz=Tgz$. Now, from Theorem 3.1(iii) it is easy to show that $fz=gfz$. Thus, $fz=gfz=Tfz=ffz$ is a common fixed point of f,T and g in X. The uniqueness part follows from Theorem 3.1(iii). □

Corollary 3.4. *Let (X,d) be a complete metric space $f,g,T:X\to X$ satisfying (i)–(iv) of Theorem 3.1 and (v) of Theorem 3.3. Then f,g, and T have a unique common fixed point in X.*

Corollary 3.5. *Let (X,d) be a complete metric space and let $\Im$ be a family of self maps of X. If there is a map T in $\Im$ such that for each pair f,g in $\Im$ satisfying (i)–(iv) of Theorem 3.1 and (v) of Theorem 3.3, then each member of $\Im$ has a unique fixed point in X which is a unique common fixed point of the family $\Im$.*

Theorem 3.6. *Let (X,d) be a complete metric space. Then $f,g,T:X\to X$ satisfying Theorem 3.1(iii) have a unique common fixed point if and only if there is $u\in X$ such that $fu=gu=Tu$ and $f^2u=g^2u=T^2u$.*

Proof. The necessary part is trivial. To prove the sufficient part, let there be a $u\in X\ni$ (a) $fu=gu=Tu$, (b) $f^2u=g^2u=T^2u$. Let $y=fu=gu=Tu$. Then from Theorem 3.1(iii) and (b), we can show that $y=fy=Ty=gy$, that is, y is a common fixed point of f,g,T in X. Further, from (iii) of Theorem 3.1, the uniqueness of y follows at once. □

Theorem 3.7. *Let X be a set and Y a Banach space. Let $f,g:X\to Y$ be such that*

(i) *$\operatorname{co}(f(X))\subset g(X)$;*

(ii) *$g(X)$ is closed in Y;*

(iii) *$\|fx-fy\|\le\varphi[\max\{\|gx-gy\|,\|gx-fx\|,\|gy-fy\|\}]$ for all $x,y\in X$ where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is nondecreasing u.s.c. function with $\varphi(qt)<t$, $1\le q\le 2$. Then there is a $u\in X$ such that $fu=gu$. Further, if f or g is injective, then u is unique.*

Proof. Choose $x\in X$. From (i) of Theorem 3.7, we define $\{x_n\}$ in X as $fx_n=gx_{n+1}$, for all $n\in\omega$. Writing $d_n=\|fx_n-fx_{n+1}\|$ and using (iii) of Theorem 3.7, we get

$$d_n<d_{n-1},\quad d_n\le\varphi(d_{n-1})\le\cdots\le\varphi^n(d_0),\quad \forall n\in\omega. \tag{3.4}$$

Now, for each $p \in \mathbb{N}$,

$$\begin{aligned}||fx_n - fx_{n+p}|| &\le \sum_{i=1}^{p-1} ||fx_{n+i} - fx_{n+1+i}|| \\ &\le \sum_{i=0}^{p-1} \varphi^{n+i}(d_0) = \varphi^n(d_0).\left\{\frac{\varphi^p(d_0)-1}{\varphi(d_0)-1}\right\} \longrightarrow 0\end{aligned} \tag{3.5}$$

as $n \to \infty$ by Lemma 2.3 implies $\{fx_n\}$ is Cauchy in Y and by assumption, $\lim_n fx_n$ exists finitely in Y. From (i), define $gy_n = afx_n + (1-a)fx_{n+1}$, $0 \le a \le 1$ in $g(X)$. We have

$$\begin{aligned}||fy_n - gy_n|| &\le a||fx_n - fy_n|| + (1-a)||fx_{n+1} - fy_n|| \\ &\le a\varphi\Big[\max\big\{||gx_n - gy_n||, ||gx_n - fx_n||, ||gy_n - fy_n||\big\}\Big] \\ &\quad + (1-a)\varphi\Big[\max\big\{||gx_{n+1} - gy_n||, ||gx_{n+1} - fx_{n+1}||, ||gy_n - fy_n||\big\}\Big].\end{aligned} \tag{3.6}$$

Also,

$$\begin{aligned}||gx_n - gy_n|| &\le ||fx_{n-1}fx_n|| + (1-a)||fx_{n-1} - fx_{n+1}|| \\ &\le \varphi^{n-1}(d_0) + (1-a)\varphi^n(d_0) \le (2-a)\varphi^{n-1}(d_0) \quad (\text{using } \varphi(d_0) < d_0), \\ ||gx_{n+1} - gy_n|| &= (1-a)||fx_n - fx_{n+1}|| \le (1-a)\varphi^n(d_0).\end{aligned} \tag{3.7}$$

Thus, from (3.4) and (3.7), (3.6) reduces to

$$\begin{aligned}||fy_n - gy_n|| &\le a\varphi\Big[\max\big\{(2-a)\varphi^{n-1}(d_0), \varphi^{n-1}(d_0), ||fy_n - gy_n||\big\}\Big] \\ &\quad + (1-a)\varphi\Big[\max\big\{(1-a)\varphi^n(d_0), \varphi^n(d_0), ||fy_n - gy_n||\big\}\Big] \\ &\le a\varphi\Big[\max\big\{(2-a)\varphi^{n-1}(d_0), ||fy_n - gy_n||\big\}\Big] \\ &\quad + (1-a)\varphi\Big[\max\big\{(2-a)\varphi^{n-1}(d_0), ||fy_n - gy_n||\big\}\Big], \\ &\qquad\qquad \text{as } \varphi(d_0) < d_0, 1 \le 2-a \le 2 \\ &\le \varphi\Big[\max\big\{(2-a)\varphi^{n-1}(d_0), ||fy_n - gy_n||\big\}\Big] \\ &\le \varphi\Big[(2-a)\varphi^{n-1}(d_0)\Big] < \varphi^{n-1}(d_0),\end{aligned} \tag{3.8}$$

otherwise, if $\|fy_n - gy_n\|$ is maximum then a contradiction arises.

Now, for any $p \in \mathbb{N}$, writing $K_p = (\varphi^p(d_0) - 1)/(\varphi(d_0) - 1)$ we get

$$\begin{aligned}||gy_n - gy_{n+p}|| &\le a||fx_n - fx_{n+p}|| + (1-a)||fx_{n+1} - fx_{n+1+p}|| \\ &\le \Big[a\varphi^n(d_0) + (1-a)\varphi^{n-1}(d_0)\Big]K_p \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \Longrightarrow \{gy_n\}\end{aligned} \tag{3.9}$$

is Cauchy in $g(X) \subset Y$, and from (ii) of Theorem 3.7 there exists $u \in X \ni \lim_n gy_n = gu$. So, from (3.4), (3.7), and (3.8) we have, $\|fx_n - fy_n\| \le \|fx_n - gx_n\| + \|gx_n - gy_n\| + \|gy_n - fy_n\| \to 0$ as $n \to \infty$. Hence, $\lim_n fx_n = \lim_n fy_n = \lim_n gy_n = \lim_n gx_n = gu$.

Now, let $fu \ne gu$. Then from (iii) of Theorem 3.7, we have $\|fu - fx_n\| \le \varphi[\max\{\|gu - gx_n\|, \|gu - fu\|, \|gx_n - fx_n\|\}]$; taking limit as $n \to \infty$, we have $\|fu - gu\| \le \varphi[\max\{0, \|fu - gu\|, 0\}] < \|fu - gu\|$ which is a contradiction. Hence $fu = gu$. The second part follows from (iii) of Theorem 3.7 and injectiveness of f or g. □

4. Coincidence point for multivalued mappings

Theorem 4.1. *Let X be a Banach space; and let $S, T : X \to \mathrm{CB}(X)$ and $f : X \to X$ be such that*

(i) $S(X)UT(X) \subseteq f(X) \in C(X)$,

(ii) *for all $x, y \in X$, $H(Sx, Ty) \le \varphi\{\|fx - fy\|, D(fx, Sx), D(fy, Ty), D(fx, Ty), D(fy, Sx)\}$ where $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ is u.s.c. and nondecreasing in each coordinate variable with $\gamma(t) = \max[\varphi(t, t, t, at, bt) : a + b = 2,\ a, b \in \{0, 1, 2\}] \le qt$, $0 \le q < 1$, $t > 0$. Then f, S and T have a coincidence point in X.*

Proof. Choose $a \in (0, 1)$ such that $q^{1-a} < 1$. Let $x_0 \in X$. Form (i), we define a sequence $\{x_n\}$ in X as $fx_{2n+1} \in Sx_{2n}$, $fx_{2n+2} \in Tx_{2n+1}$ such that

$$\begin{aligned} ||fx_{2n+1} - fx_{2n+2}|| &< q^{-a} H(Sx_{2n}, Tx_{2n+1}), \\ ||fx_{2n+2} - fx_{2n+3}|| &< q^{-a} H(Tx_{2n+1}, Sx_{2n+2}), \end{aligned} \tag{4.1}$$

for all $n \in \omega$, writing $d_n = \|fx_n - fx_{n+1}\|$, we have from (ii) by routine calculations that $d_{2n+1} \le d_{2n}$ and $d_{2n+1} \le q^{1-a} d_{2n}$. Similarly, $d_{2n+2} \le d_{2n+1}$ and $d_{2n+2} \le q^{1-a} d_{2n+1}$. Thus, combining these we can write

$$d_{n+1} \le d_n, \quad d_n \le q^{1-a} d_{n-1} \le \cdots \le q^{(1-a)n} d_0, \quad \forall n \in \omega, 0 \le q^{1-a} < 1. \tag{4.2}$$

This shows that $\{fx_n\}$ is Cauchy in $f(X)$ and from (i) of Theorem 4.1, there exists $z \in X \ni \lim fx_n = fz$,

$$\begin{aligned} D(fz, Sz) &\le ||fz - fx_{2n+2}|| + D(fx_{2n+2}, Sz) \le ||fz - fx_{2n+2}|| + H(Sz, Tx_{2n+1}) \\ &\le \varphi\Big\{||fz - fx_{2n+1}||, D(fz, Sz), D(fx_{2n+1}, Tx_{2n+1}), D(fz.Tx_{2n+1}), \\ &\qquad D(fx_{2n+1}, Sz)\Big\} + ||fz - fx_{2n+2}|| \\ &\le \varphi\Big\{||fz - fx_{2n+1}||, D(fz, Sz), ||fx_{2n+1} - fx_{2n+2}||, ||fz - fx_{2n+2}||, \\ &\qquad \Big(||fx_{2n+1} - fz|| + D(fz, Sz)\Big)\Big\} + ||fz - fx_{2n+2}||. \end{aligned} \tag{4.3}$$

As $n \to \infty$, we have $D(fz, Sz) \le \varphi\{0, D(fz, Sz), 0, 0, D(fz, Sz)\} \le \varphi\{t, t, t, t, t\} \le qt$ (where $t = D(fz, Sz)$) which implies that $fz \in \overline{Sz} = Sz$. Similarly $fz \in Tz$.

Hence z is a coincidence point of f, S and T in X. □

In [3, Theorem 2] the continuity of the involved maps are taken; but in Theorem 4.1 instead of the continuity condition of the maps we take only $f(X) \in C(X)$ for the existence of a coincidence point; to support this we give the following example.

Example 4.2. Let $X = [0,1]$. Define $S,T : X \to \text{CB}(X)$ and $f : X \to X$ as follows:

$$Sx = \begin{cases} \{0\}, & 0 \le x \le \frac{1}{2}, \\ \left\{\frac{1}{4}\right\}, & \frac{1}{2} < x \le 1, \end{cases} \qquad Tx = \begin{cases} \{0\}, & 0 \le x \le \frac{1}{2}, \\ \left\{\frac{1}{4}\right\}, & \frac{1}{2} < x \le 1, \end{cases} \qquad fx = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2} < x < 1, \\ \frac{2}{3}, & x = 1. \end{cases} \tag{4.4}$$

Then $SX = \{0,1/4\} = TX$, $fX = \{0,1/4,2/3\} \in C(X)$; S,T, and f are discontinuous. Let $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ be given by $\varphi(t_1,t_2,t_3,t_4,t_5) = \sqrt{t_1}/2$, $t_i > 0$; then $\gamma(t) = \sqrt{t_1}/2$. Clearly S,T,f and φ, γ satisfy all the conditions of Theorem 4.1 with $q = 1/2$ and $0 = f0 \in S0 = T0$, that is, 0 is a coincidence point of S,T, and f.

Theorem 4.3. *Let X be a Banach space and $f : X \to X$, $S,T : X \to C(X)$ satisfy (i)–(ii) of Theorem 4.1 and (iii) (f,S) and (f,T) are compatible pairs. Then there is a point $z \in X$ such that $fz \in Sz \cap Tz$. Suppose that $\{z_n = f^n z\}$ is a sequence of iterate in X for z and $\{S_n\},\{T_n\}$ are sequences of multifunctions on X where $S_n z = Sf^{n-1}z$,*

$$T_n z = Tf^{n-1}z, \quad f^n z \in S_n z \cap T_n z, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

If $z_n \to z$ and $\{S_n\},\{T_n\}$ converge, respectively, to S and T on X pointwise, then z is a common fixed point of S and T.

Proof. From Theorem 4.1, there is $z \in X \ni fz \in Sz \cap Tz$. Again from (ii) of Theorem 4.1, it is easy to show that $Sz = Tz$. Again, from (iii) of Theorem 4.3 and Lemma 2.4, we have $fz \in Sz = Tz \Rightarrow f^2 z \in fSz = Sfz$, $f^2 z \in fTz = Tfz$, and $Sfz = Tfz$. Continuing this process, we get $S_n z = Sf^{n-1}z = Tf^{n-1}z = T_n z$ where $z_n = f^n z \in Sf^{n-1}z = Tf^{n-1}z$. By hypothesis, $S_n z \to Sz$ and $T_n z \to Tz$. Then

$$\begin{aligned} D(z,Sz) &\le \|z - z_n\| + D(z_n,Sz) \\ &\le \|z - z_n\| + H(S_n z,Sz) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \text{ which implies that } z \in \overline{Sz} = Sz. \end{aligned} \tag{4.6}$$

As $Sz = Tz$, hence z is a common fixed point of S and T in X. □

Acknowledgement. The authors convey their gratitudes to the referee for his valuable suggestions.

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