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ON THE SHARP CONSTANT FOR STARLIKENESS

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ABSTRACT. We obtain a sharp constant of the sufficient condition for p-valently starlikeness, which had been studied by Nunokawa (1991), Obradović and Owa (1989), and Li (1993).

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1. Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic in $\mathbf{U} = \{z : |z| < 1\}$. A function $f(z)$ in $A(p)$ is said to be p-valently starlike if and only if

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{1.2}$$

Let $S(p)$ denote the subclass of $A(p)$ consisting of all functions $f(z)$ which are p-valently starlike in $\mathbf{U}$ (cf. [1]). For a function $g(z)$ in $A(p)$, the interesting problem is to find the best constant A such that $g(z)$ is in $S(p)$ whenever

$$\left|1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)}\right| < A\left|\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right| \quad \text{in } \mathbf{U}. \tag{1.3}$$

In 1989, Obradović and Owa [6] obtained that $A = 5/4$ for the case of $p = 1$. For the general case, Nunokawa [5] gained that $A = \log 4$. Recently, Li [2] improved these results and obtained that $A = 3/2$. In this paper, we will solve this problem completely and give the sharp constant $A = 1.80898\ldots$, where A is the unique solution of the equation

$$xe^{1/(x^2-1)} = x + 1. \tag{1.4}$$

For proving our result, we should recall the concept of subordination between analytic functions. Given two analytic functions $f(z)$ and $F(z)$, the function $f(z)$ is said to be subordinate to $F(z)$ if $F(z)$ is univalent in $\mathbf{U}$, $f(0) = F(0)$, and $f(\mathbf{U}) \subset F(\mathbf{U})$. We denote this subordination by $f(z) \prec F(z)$ (see [7]).

Suppose that $h(z)$ is analytic in $\mathbf{U}$, and that $\Phi(z)$ is analytic in an appropriate domain $\mathbf{D}$, we consider the following first-order differential subordination

$$\beta + zp'(z)\Phi(p(z)) \prec h(z), \tag{1.5}$$

where $p(z)$ is analytic in $\mathbf{U}$, β is a complex constant. Changing the "$\prec$" of (1.5) to "$=$", we get the corresponding first-order differential equation

$$\beta + zp'(z)\Phi(p(z)) = h(z). \tag{1.6}$$

2. Main results. Our results rest on the following lemma, which is the special case of [3, Theorem 3].

Lemma 2.1. *Suppose that $h(z)$ is a starlike function in $\mathbf{U}$, $\Phi(z)$ is analytic in the domain $\mathbf{D}$ and $p(z)$, $q(z)$ are two analytic functions in $\mathbf{U}$. If $p(z)$ satisfies the relation (1.5), $q(z)$ is a univalent solution of the corresponding equation (1.6) and $p(0) = q(0)$, then $p(z) \prec q(z)$.*

Theorem 2.2. *Let $g(z) \in A(p)$, and suppose that*

$$\left|1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)}\right| < A\left|\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right| \quad \text{in } \mathbf{U}, \tag{2.1}$$

where the constant A is given by (1.4). Then $g(z) \in S(p)$ and the result is sharp.

Proof. Let

$$f(z) = \frac{g^{(p-1)}(z)}{p!}. \tag{2.2}$$

Then $f(z) \in A(1)$. From the assumption (2.1), $f(z)$ satisfies

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < A\left|\frac{zf'(z)}{f(z)}\right| \quad \text{in } \mathbf{U}. \tag{2.3}$$

By putting $p(z) = zf'(z)/f(z)$, equation (2.3) can be rewritten as

$$\left|1 + \frac{zp'(z)}{p^2(z)}\right| < A. \tag{2.4}$$

Let $\varphi(z) = A(1+Az)/(A+z)$ for $z \in \mathbf{U}$. Obviously $\varphi(z)$ is a conformal mapping from $\mathbf{U}$ to $\Omega = \{w : |w| < A\}$ and $\varphi(0) = 1$. Combining (2.4) with the definition of subordination, we obtain

$$1 + \frac{zp'(z)}{p^2(z)} \prec \frac{A(1+Az)}{A+z}. \tag{2.5}$$

Setting

$$q(z) = \frac{1}{1 + (A^2-1)\log A/(A+z)}, \tag{2.6}$$

we have

$$1 + \frac{zq'(z)}{q^2(z)} = \frac{A(1+Az)}{A+z} \tag{2.7}$$

and $p(0) = q(0) = 1$. As $A > 1$, we can choose a uniform analytic branch of $\log(A+z)$ such that $q(z)$ is univalent on this branch. By taking the real part of the denominator of $q(z)$ and combining (1.4), we conclude that

$$\Re\left[1 + (A^2-1)\log\frac{A}{A+z}\right] > 1 + (A^2-1)\log\frac{A}{A+1} = 0. \tag{2.8}$$

It follows that $\Re[q(z)] > 0$, so $q(z)$ is analytic and univalent. Let $\mathbf{D} = \mathbb{C}\backslash\{0\}$, $\Phi(z) = 1/z^2$, $\beta = 1$, and $h(z) = A(1+Az)/(A+z)$, where $\mathbb{C}$ is the complex plane. It is clear that $h(z)$ is a starlike function. From Lemma 2.1, we deduce that $p(z) \prec q(z)$. Hence

$$\Re\left[\frac{zf'(z)}{f(z)}\right] = \Re[p(z)] \geq \min_{|z|=r<1} \Re[q(z)] > 0. \tag{2.9}$$

This is equivalent to

$$\Re\left[\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right] = \Re\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{2.10}$$

From [4, Theorem 5], we have

$$\Re\left[\frac{zg'(z)}{g(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{2.11}$$

This proves $g(z) \in S(p)$.

For any $A_1 > A = 1.80898\ldots$, we get a function $q_1(z)$ by replacing A in (2.6) with A_1 and choosing an appropriate branch of $\log(A_1+z)$. We can easily observe that the real part of $q_1(z)$ is not always positive. Through the relations $q_1(z) = zf'(z)/f(z)$ and $f(z) = g^{(p-1)}(z)/p!$, we can construct an analytic function $g(z)$ which belongs to $A(p)$ and satisfies (2.1), but it is not in $S(p)$. This completes the proof. □

Taking $p = 1$ in Theorem 2.2, we easily have the following corollary.

Corollary 2.3. *If $f(z) \in A(1)$ and it satisfies the condition*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < A\left|\frac{zf'(z)}{f(z)}\right| \quad \textit{in } \mathbf{U}, \tag{2.12}$$

where the constant A is given by (1.4), then $f(z)$ is univalent and starlike in **U**.

The problem that Nunokawa proposed in [5] has been solved completely, but the converse proposition of Theorem 2.2 is not true. We find a simple example $f(z) = z/(1-z)$ which belongs to $S(1)$, but it does not satisfy (2.12). The following theorem is better than (2.1) because it includes at least this example.

Theorem 2.4. *Let $g(z) \in A(p)$, and suppose that*

$$\left|1 + \frac{zg^{(p+1)}(z)}{g^{(p)}(z)} - \frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right| < \left|\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right| \quad \textit{in } \mathbf{U}. \tag{2.13}$$

Then $g(z) \in S(p)$.

Proof. Let

$$f(z) = \frac{g^{(p-1)}(z)}{p!}. \tag{2.14}$$

Then $f(z) \in A(1)$. From the assumption (2.13), $f(z)$ satisfies

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \left|\frac{zf'(z)}{f(z)}\right| \quad \text{in } \mathbf{U}. \tag{2.15}$$

By setting $p(z) = zf'(z)/f(z)$, equation (2.15) can be rewritten as

$$\left|\frac{zp'(z)}{p^2(z)}\right| < 1. \tag{2.16}$$

From the definition of subordination, we obtain

$$\frac{zp'(z)}{p^2(z)} \prec z. \tag{2.17}$$

Let $q(z) = 1/(1-z)$, we observe that $zq'(z)/q^2(z) = z$, $p(0) = q(0) = 1$, and $\Re[q(z)] > 0$. From Lemma 2.1, we know that $p(z) \prec 1/(1-z)$. Therefore

$$\Re\left[\frac{zf'(z)}{f(z)}\right] = \Re[p(z)] \geq \min_{|z|=r<1} \Re[q(z)] > 0. \tag{2.18}$$

This is equivalent to

$$\Re\left[\frac{zg^{(p)}(z)}{g^{(p-1)}(z)}\right] = \Re\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{2.19}$$

From [4, Theorem 5], we have

$$\Re\left[\frac{zg'(z)}{g(z)}\right] > 0 \quad \text{in } \mathbf{U}. \tag{2.20}$$

This completes the proof. □

Taking $p = 1$ in Theorem 2.4, we obviously have the following corollary.

Corollary 2.5. *If $f(z) \in A(1)$ and it satisfies the condition*

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| < \left|\frac{zf'(z)}{f(z)}\right| \quad \textit{in } \mathbf{U}, \tag{2.21}$$

then $f(z) \in S(1)$.

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References

[1] A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. **68** (1950), 204–223. MR 11,508d. Zbl 037.05502.

[2] J. L. Li, *On a criterion of starlikeness*, Math. Japon. **38** (1993), no. 5, 897–899. MR 94j:30012. Zbl 786.30010.

[3] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J. **32** (1985), no. 2, 185–195. MR 86h:30046. Zbl 575.30019.

[4] M. Nunokawa, *On the theory of multivalent functions*, Tsukuba J. Math. **11** (1987), no. 2, 273–286. MR 89d:30013. Zbl 639.30014.

[5] ______, *On certain multivalent functions*, Math. Japon. **36** (1991), no. 1, 67–70. MR 92b:30017. Zbl 718.30010.

[6] M. Obradović and S. Owa, *A criterion for starlikeness*, Math. Nachr. **140** (1989), 97–102. MR 90i:30020. Zbl 676.30009.

[7] C. Pommerenke, *Univalent Functions.* With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, vol. 25, Vandenhoeck & Ruprecht, Göttingen, 1975. MR 58#22526. Zbl 298.30014.

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