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A REMARK ON GWINNER'S EXISTENCE THEOREM ON VARIATIONAL INEQUALITY PROBLEM

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ABSTRACT. Gwinner (1981) proved an existence theorem for a variational inequality problem involving an upper semicontinuous multifunction with compact convex values. The aim of this paper is to solve this problem for a multifunction with open inverse values.

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1. Introduction. In 1981, Gwinner [1] proved an existence theorem for a variational inequality problem, which is an infinite dimensional version of Walras excess demand theorem (see also Zeidler [5]).

Theorem 1.1. *Let P and Q be nonempty compact convex subsets of locally convex Hausdorff topological vector spaces X and Y, respectively. Let $f : P \times Q \to \mathbb{R}$ be continuous. Let $S : P \to Q$ be a multifunction. Suppose that*
- (i) *for each $y \in Q, \{x \in P : f(x,y) < t\}$ is convex for all $t \in \mathbb{R}$,*
- (ii) *S is an upper semicontinuous multifunction with nonempty compact convex values. Then there exist $x_0 \in P$ and $y_0 \in S(x_0)$ such that $f(x_0,y_0) \le f(x,y_0)$ for all $x \in P$.*

In this paper, our aim is to obtain the above variational inequality for a multifunction with open inverse values. We follow the method of Tarafdar and Yuan [4].

2. Preliminaries. In $N \in \mathbb{N}$, let $\langle N \rangle$ be the set of all nonempty subsets of $\{0,1,2,\dots,N\}$, $\Delta_N = \operatorname{co}\{e_0,e_1,\dots,e_N\}$ be the standard simplex of dimension N, where $\{e_0,e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^{N+1}$, and for $J \in \langle N \rangle$, let $\Delta_J = \operatorname{co}\{e_j : j \in J\}$. Horvath [2] proved the following result.

Lemma 2.1. *Let X be a topological space and $F : \langle N \rangle \to X$. For each $J \in \langle N \rangle$, let $F(J)$ be a nonempty contractible subset of X and for all $J, J' \in \langle N \rangle$ such that $J \subseteq J'$, suppose that $F(J) \subseteq F(J')$. Then there exists a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subset F(J)$ for all $J \in \langle N \rangle$.*

Also, we need the following fixed point theorem due to Lassonde [4].

Lemma 2.2. *Let $F : \Delta_N \to \Delta_N$ be a multifunction such that $F = F_n \circ F_{n-1} \circ \cdots \circ F_1 \circ F_0$, $\Delta_N \xrightarrow{F_0} X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} \cdots \xrightarrow{F_n} X_{n+1} = \Delta_N$, where each F_i is either a single-valued continuous function (in which case X_{i+1} is assumed to be a Hausdorff topological space)* *or an upper semicontinuous multifunction with $F_i(x)$, a nonempty compact convex subset of X_{i+1} (in which case X_{i+1} is a convex subset of a Hausdorff topological vector space). Then F has a fixed point.*

3. Main theorem

Theorem 3.1. *Let P as in Theorem 1.1 and Q be an arbitrary subset of a locally convex Hausdorff topological vector space Y. Let $f : P\times Q \to \mathbb{R}$ be continuous and satisfy condition (i) of Theorem 1.1. Let $S : P \to Q$ be a multifunction such that*

(i) *$S^{-1}(X)$ is open for all $x \in Q$;*

(ii) *for each open set $F \subset P$, the set $\bigcap_{y\in F} S(y)$ is empty or contractible;*

(iii) *$S(P)$ is compact and contractible. Then the conclusion of Theorem 1.1 holds.*

Proof. Since P is compact, there exists a finite subset $\{x_0,x_1,x_2,\dots,x_N\}$ of $S(P)$ such that $P = \bigcup_{i=0}^{N} S^{-1}(x_i)$. Define $F : \langle N\rangle \to S(P)$ by

$$F(J) = \begin{cases} \bigcap\left\{S(y) : y \in \bigcap_{j\in J} S^{-1}(x_j)\right\} & \text{if } \bigcap_{j\in J} S^{-1}(x_j) \neq \varnothing, \\ S(P) & \text{otherwise.} \end{cases} \tag{3.3.1}$$

It is clear that if $y \in \bigcap_{j\in J} S^{-1}(x_j)$, then $x_j \in S(y)$ for all $j \in J$. Thus, $F(J)$ is nonempty and contractible. Further, $F(J) \subseteq F(J')$ whenever $J \subseteq J'$. By Lemma 2.1, there exists a continuous function $f : \Delta_N \to S(K)$ such that $f(\Delta_J) \subset F(J)$ for all $J \in \langle N\rangle$. Let $\{g_i : i \in \{0,1,2,\dots,N\}\}$ be a continuous partition of unity subordinated to the covering $\{S^{-1}(x_i) : i \in \{0,1,\dots,N\}\}$, that is, for each i, $g_i : P \to [0,1]$ is continuous, $\{y \in P : g_i(y) \neq \varnothing\} \subset S^{-1}(x_i)$, and $\sum_{i=0}^{N} g_i(y) = 1$ for all $y \in P$. Now, define $g : P \to \Delta_N$ by $g(y) = (g_0(y), g_1(y),\dots,g_N(y))$ for all $y \in P$. Then g is continuous. Further, $g(y) \in \Delta_{J(y)}$ for all $y \in P$, where $J(y) = \{i : g_i(y) \neq 0\}$. Therefore, $f\circ g(y) \in f(\Delta_J(y)) \subset F_{J(y)} \subset S(y)$.

Consider $T : S(P) \to P$ defined by $T(y) = \{z \in P : f(z,y) \le f(w,y)$ for all $w \in p\}$. For each $y \in S(P)$, $T(y)$ is nonempty since f assumes its minimum on the compact set P. Also, it is closed and hence compact. Further, $T(y)$ is convex. Indeed, let z_1 and $z_2 \in P$ be such that $f(z_i,y) \le f(w,y)$ for all $w \in P$ and $i = 1,2$. By the assumption on f, $f(\lambda z_1 + (1-\lambda)z_2, y) \le f(w,y)$ for all $w \in P$. Since f is continuous, the graph of T, $\mathrm{Gr}(T) = \{(y,z) : y \in S(P),\ z \in T(y)\}$ is a closed subset of the compact set $S(P)\times P$. Then it follows that T is upper semicontinuous.

Consider $G := g\circ T\circ f : \Delta_N \to \Delta_N$. Now, by Lemma 2.2, there exists $z_0 \in \Delta_N$ such that $z_0 \in G(z_0)$. Let $y_0 = f(z_0)$. Then $y_0 \in f\circ g\circ T\circ f(z_0)$, that is, there exists $x_0 \in T(y_0)$ so that $y_0 \in f\circ g(x_0) \in S(x_0)$. This completes the proof. □

References

[1] J. Gwinner, *On fixed points and variational inequalities—a circular tour*, Nonlinear Anal., Theory Methods Appl. **5** (1981), no. 5, 565–583. MR 82e:49018. Zbl 461.47037.

[2] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), no. 2, 341–357. MR 92f:52002. Zbl 733.54011.

[3] M. Lassonde, *Fixed points for Kakutani factorizable multifunctions*, J. Math. Anal. Appl. **152** (1990), no. 1, 46–60. MR 91h:47062. Zbl 719.47043.

[4] E. Tarafdar and X. Z. Yuan, *A remark on coincidence theorems*, Proc. Amer. Math. Soc. **122** (1994), no. 3, 957–959. MR 95a:47066. Zbl 818.47056.

[5] E. Zeidler, *Nonlinear Functional Analysis and its Applications. III*, Variational methods and optimization. Translated from the German by Leo F. Boron., Springer-Verlag, New York, Berlin, 1985. MR 90b:49005. Zbl 583.47051.

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