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LEFT GLOBAL DIMENSIONS AND INVERSE POLYNOMIAL MODULES

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ABSTRACT. We prove the fact $\text{l.gl.dim}\, R[x] = (\text{l.gl.dim}\, R) + 1$, where l. gl. dim means the left global dimension by using inverse polynomial modules and injective dimensions. The classical way to prove the fact $\text{l.gl.dim}\, R[x] = (\text{l.gl.dim}\, R) + 1$ is using polynomial modules and projective dimensions.

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1. Introduction. The classical way to prove the fact $\text{l.gl.dim}\, R[x] = (\text{l.gl.dim}\, R) + 1$, where l. gl. dim means the left global dimension is using the construction $M[x]$ (where M is any left R-module). In this paper, we give another proof of this fact by using inverse polynomial module $M[x^{-1}]$ and injective dimensions instead of polynomial module $M[x]$ and projective dimensions. Northcott [3] and McKerrow [1] showed that the polynomial module $M[x]$ and the inverse polynomial module $M[x^{-1}]$ are not isomorphic as left $R[x]$-modules by showing that if R is a left noetherian ring and $E \neq 0$ is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module while $E[x]$ is not an injective left $R[x]$-module. Park in [5] also showed that if $P \neq 0$ is a projective left R-module, then $P[x^{-1}]$ is not a projective left $R[x]$-module while $P[x]$ is a projective left $R[x]$-module. Inverse polynomial modules were developed in [1, 3, 4, 5], and recently in [2].

Definition 1.1. Let R be a ring and M be a left R-module, then $M[x^{-1}]$ is a left $R[x]$-module such that

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1} \tag{1.1}$$

and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}, \tag{1.2}$$

where $r \in R$. Similarly, we can define $M[[x^{-1}]]$ as a left $R[x]$-module.

Lemma 1.2. *If $E[[x^{-1}]]$ is an injective left $R[x]$-module, then E is an injective left R-module.*

Proof. Let $I \subset R$ be a left ideal and let $f : I \to E$ be a R-linear map. Then the map $I[[x^{-1}]] \to E[[x^{-1}]] (\sum_{i=0}^{\infty} r_i x^{-i} \to \sum_{i=0}^{\infty} f(r_i) x^{-i})$ is an $R[x]$-linear map so can be extended to a map

$$R[[x^{-1}]] \xrightarrow{g} E[[x^{-1}]]. \tag{1.3}$$

Since $xR = 0$ (for $R \subset R[[x^{-1}]]$), $xg(R) = 0$ in $E[[x^{-1}]]$. But this implies $g(R) \subset E$ (with $E \subset E[[x^{-1}]]$). Then the map $R \to E$ agreeing with g is an R-linear map and extends $I \to E$. So E is an injective R-module. □

Theorem 1.3. *Let M be a left R-module, then*

$$\operatorname{inj\,dim}_{R[x]}\left(M[[x^{-1}]]\right) = \operatorname{inj\,dim}_R(M). \tag{1.4}$$

Proof. Let R be a ring and E be an injective left R-module. Define $\phi : \operatorname{Hom}_R(R[x],E) \to E[[x^{-1}]]$ by $\phi(f) = f(1) + f(x)x^{-1} + f(x^2)x^{-2} + \cdots$, then $\operatorname{Hom}_R(R[x],E)$ and $E[[x^{-1}]]$ are isomorphic as left left $R[x]$-modules. Since $\operatorname{Hom}_R(R[x],E)$ is an injective left $R[x]$-module so is $E[[x^{-1}]]$. Suppose $\operatorname{inj\,dim}_R(M) = n$ and

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow 0 \tag{1.5}$$

is an injective resolution of M. Then

$$0 \longrightarrow M[[x^{-1}]] \longrightarrow E^0[[x^{-1}]] \longrightarrow E^1[[x^{-1}]] \longrightarrow \cdots \longrightarrow E^n[[x^{-1}]] \longrightarrow 0 \tag{1.6}$$

is an injective resolution of $M[[x^{-1}]]$. Let

$$K^i = \ker\left(E^i \longrightarrow E^{i+1}\right) \quad \text{for } 0 \le i < n. \tag{1.7}$$

Then K^i is not an injective R-module for $0 \le i < n$. So by the above lemma $K^i[[x^{-1}]]$ is not an injective $R[x]$-module. So then we get

$$\operatorname{inj\,dim}_{R[x]}\left(M[[x^{-1}]]\right) = n. \tag{1.8}$$

Suppose $\operatorname{inj\,dim}_R(M) = \infty$ and

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^n \longrightarrow \cdots \tag{1.9}$$

is an injective resolution of M. Then

$$0 \longrightarrow M[[x^{-1}]] \longrightarrow E^0[[x^{-1}]] \longrightarrow E^1[[x^{-1}]] \longrightarrow \cdots \longrightarrow E^n[[x^{-1}]] \longrightarrow \cdots \tag{1.10}$$

is an injective resolution of $M[[x^{-1}]]$. But K^i is not an injective R-module for all i. Thus $K^i[[x^{-1}]]$ is not an injective $R[x]$-module for all i. Therefore, $\operatorname{inj\,dim}_{R[x]}(M[[x^{-1}]]) = \infty$. Similarly, if

$$\operatorname{inj\,dim}_{R[x]}\left(M[[x^{-1}]]\right) = n, \quad \text{then } \operatorname{inj\,dim}_R(M) = n \tag{1.11}$$

and if

$$\operatorname{inj\,dim}_{R[x]}\left(M[[x^{-1}]]\right) = \infty, \quad \text{then } \operatorname{inj\,dim}_R(M) = \infty. \tag{1.12}$$

Hence, $\operatorname{inj\,dim}_{R[x]}(M[[x^{-1}]]) = \operatorname{inj\,dim}_R(M)$. □

Theorem 1.4. *Let M, $M[[X^{-1}]]$ be left $R[x]$-modules. Then there is a short exact sequence of $R[x]$-modules*

$$0 \longrightarrow M \longrightarrow M[[x^{-1}]] \longrightarrow M[[x^{-1}]] \longrightarrow 0. \tag{1.13}$$

Proof. Let $\phi : M \to [[x^{-1}]]$ be defined by

$$\phi(y) = y + (xy)x^{-1} + (x^2y)x^{-2} + (x^3y)x^{-3} + \cdots, \quad \text{for } y \in M, \tag{1.14}$$

then ϕ is an injective $R[x]$-linear map. Let $\psi : M[[x^{-1}]] \to M[[x^{-1}]]$ be defined by

$$\psi(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots) = (m_1 - xm_0) + (m_2 - xm_1)x^{-1} + \cdots, \tag{1.15}$$

then ψ is a surjective $R[x]$-linear map. Let y be an element of M, then

$$\begin{aligned}(\psi \circ \phi)(y) &= \psi(y + (xy)x^{-1} + (x^2y)x^{-2} + \cdots) \\ &= (xy - xy) + (x^2y - x^2y)x^{-1} + (x^3y - x^3y)x^{-2} + \cdots = 0.\end{aligned} \tag{1.16}$$

Therefore, image $(\phi) \subset \ker(\psi)$. Let $m_0 + m_1x^{-1} + m_2x^{-2} + \cdots$ be an element of $\ker(\psi)$, then

$$\begin{aligned}&\psi(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots) \\ &\quad = (m_1 - xm_0) + (m_2 - xm_1)x^{-1} + (m_3 - xm_2)x^{-2} + \cdots = 0.\end{aligned} \tag{1.17}$$

Therefore, $m_{i+1} = xm_i$, for all $i = 0, 1, 2, \ldots$. Then, for $m_0 \in M$,

$$\begin{aligned}\phi(m_0) &= m_0 + (xm_0)x^{-1} + (x^2m_0)x^{-2} + \cdots \\ &= m_0 + m_1x^{-1} + m_2x^{-2} + \cdots, \quad \text{since } m_{i+1} = xm_i.\end{aligned} \tag{1.18}$$

So, $m_0 + m_1x^{-1} + \cdots \in$ image (ϕ). Thus, $\ker(\psi) \subset$ image (ϕ). Therefore, image $(\phi) = \ker(\psi)$. Hence, $0 \to M \to M[[x^{-1}]] \to M[[x^{-1}]] \to 0$ is a short exact sequence of $R[x]$-modules. □

Corollary 1.5. $\text{l.gl.dim}\,R \le \text{l.gl.dim}\,R[x] \le \text{l.gl.dim}\,R + 1$.

Proof. The first inequality follows from Theorem 1.3. The second follows from Theorems 1.4 and 1.3. □

2. Main theorem

Theorem 2.1. $\text{l.gl.dim}\,R[x] = (\text{l.gl.dim}\,R) + 1$.

Proof. From Corollary 1.5 we see that we only need argue that if $M \neq 0$ is a left R-module and if $\text{inj}\dim M = n < \infty$, then there is a left $R[x]$-module N with $\text{inj}\dim N = n + 1$. In fact we show that M itself can be made into such an $R[x]$-module. Let $M \neq 0$ be a left R-module and let $\text{inj}\dim M = n < \infty$, and make M into an $R[x]$-module with $xM = 0$. By induction on n. Consider the short exact sequence of $R[x]$-modules

$$0 \longrightarrow M \longrightarrow M[[x^{-1}]] \longrightarrow M[[x^{-1}]] \longrightarrow 0. \tag{2.1}$$

From this short exact sequence we see that $\operatorname{inj\,dim}_{R[x]} M \le n+1$, since $\operatorname{inj\,dim}_{R[x]} M[[x^{-1}]] = \operatorname{inj\,dim}_R M = n$. So we only need to prove that for any n, if $\operatorname{inj\,dim}_R M = n$, then $\operatorname{inj\,dim}_{R[x]} M > n$. If $n = 0$, then M is an injective left R-module. But any injective $R[x]$-module is x-divisible. Also $xM = 0$. So M is not x-divisible. Hence, M is not an injective $R[x]$-module. Now suppose $n = 1$. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow \frac{E}{M} \longrightarrow 0 \tag{2.2}$$

with E and E/M injective left R-modules. Since M is a submodule of $E[[x^{-1}]]$ which is an injective left $R[x]$-module as left $R[x]$-module, we have the short exact sequence of $R[x]$-modules

$$0 \longrightarrow M \longrightarrow E[[x^{-1}]] \longrightarrow \frac{E[[x^{-1}]]}{M} \longrightarrow 0. \tag{2.3}$$

Now we want to argue $E[[x^{-1}]]/M$ is not an injective left $R[x]$-module. Suppose $E[[x^{-1}]]/M$ is an injective left $R[x]$-module. Let $I = (x) \subset R[x]$. Consider the submodule of all z in $E[[x^{-1}]]/M$ such that $xz = 0$. Then this submodule of $E[[x^{-1}]]/M$ is isomorphic to $E/M \oplus Mx^{-1}$ as an $R[x]/(x) \cong R$ module, i.e., is isomorphic to $E/M \oplus M$. So if $E[[x^{-1}]]/M$ were an injective left $R[x]$-module, then $E/M \oplus M$, and so M, would be an injective left R-module. Therefore by this contradiction we see that $E[[x^{-1}]]/M$ is not an injective left $R[x]$-module. Now we suppose $\operatorname{inj\,dim} M = n > 1$ and make the induction hypothesis. Let

$$0 \longrightarrow M \longrightarrow E \longrightarrow C \longrightarrow 0 \tag{2.4}$$

be an exact sequence of left R-modules with E an injective left R-module. Then $\operatorname{inj\,dim} C = n-1$. Make this into an exact sequence of left $R[x]$-modules with $xM = 0$, $xE = 0$, and $xC = 0$. Then by the induction hypothesis $\operatorname{inj\,dim}_{R[x]} E = 1$, $\operatorname{inj\,dim}_{R[x]} C = (n-1)+1 = n$. Then since $n > 1$ we get that $\operatorname{inj\,dim}_{R[x]} M = n+1$. This implies $\operatorname{l.gl.dim} R[x] = (\operatorname{l.gl.dim} R) + 1$. □

References

[1] A. S. McKerrow, *On the injective dimension of modules of power series*, Quart. J. Math. Oxford Ser. (2) **25** (1974), 359–368. MR 51#8098. Zbl 302.16027.

[2] L. Melkersson, *Content and inverse polynomials on Artinian modules*, Comm. Algebra **26** (1998), no. 4, 1141–1145. MR 98k:16030. Zbl 931.13005.

[3] D. G. Northcott, *Injective envelopes and inverse polynomials*, J. London Math. Soc. (2) **8** (1974), 290–296. MR 50#13003. Zbl 284.13012.

[4] S. Park, *Inverse polynomials and injective covers*, Comm. Algebra **21** (1993), no. 12, 4599–4613. MR 94k:16008. Zbl 794.16004.

[5] ———, *The Macaulay-Northcott functor*, Arch. Math. (Basel) **63** (1994), no. 3, 225–230. MR 95f:16004. Zbl 804.18009.

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