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DERIVATIONS OF CERTAIN OPERATOR ALGEBRAS

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ABSTRACT. Let $\mathcal{N}$ be a nest and let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of alg$\mathcal{N}$. We give several conditions under which any derivation δ from $\mathcal{A}$ into $L(H)$ must be inner. The conditions include (1) $H_- \neq H$, (2) $0_+ \neq 0$, (3) there is a nontrivial projection in $\mathcal{N}$ which is in $\mathcal{A}$, and (4) δ is norm continuous. We also give some applications.

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1. Introduction. In this paper, we unify some results on derivations by considering derivations from an algebra $\mathcal{A}$ containing all rank one operators of a nest algebra into an $\mathcal{A}$-bimodule $\mathcal{B}$. Chernoff [1] proves that every derivation from $F(H)$ into $L(H)$ is inner. In [2], Christensen proves that every derivation from a nest algebra into itself or into $L(H)$ is inner. In [3], Christensen and Peligrad show that every derivation of a quasitriangular operator algebra into itself is inner. Knowles [7] generalizes the result of [2] and gets that any derivation from a nest algebra into an ideal $\mathcal{J}$ of $L(H)$ is inner. Let $\mathcal{N}$ be a nest of subspaces of a Hilbert space H, let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of alg$\mathcal{N}$, and let δ be a derivation from $\mathcal{A}$ into $L(H)$. We prove that if one of the following conditions holds:

(1) $H_- \neq H$,
(2) $0_+ \neq 0$,
(3) there exists a nontrivial $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, then δ is inner.

We also prove that for any nest, if δ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then δ is inner.

We discuss some applications of these results.

Let H be a complex separable Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $K(H)$ the ideal of all compact operators in $L(H)$, $F(H)$ the subalgebra of all finite rank operators on H, and $F_1(H)$ the subset of all operators in $F(H)$ with rank less than or equal to 1. We call a subalgebra $\mathcal{A}$ of $L(H)$ *standard* provided $\mathcal{A}$ contains $F(H)$. A collection $\mathcal{L}$ of subspaces of H is said to be a *subspace lattice* if it contains zero and H, and is complete in the sense that it is closed under the formation of arbitrary closed linear spans and intersections. A subspace lattice $\mathcal{N}$ is called a *nest* if it is a totally ordered subspace lattice. Given a nest $\mathcal{N}$, let alg$\mathcal{N} = \{T \in L(H) : TN \subseteq N,\ N \in \mathcal{N}\}$. Alg$\mathcal{N}$ is said to be the *nest algebra* associated with $\mathcal{N}$. If $\mathcal{N}$ is a nest and $E \in \mathcal{N}$, then we define $E_- = \vee\{F \in \mathcal{N} : F \subsetneq E\}$, and $E_+ = \wedge\{F \in \mathcal{N} : F \supsetneq E\}$. If $e,\ f \in H$ we write $e^* \otimes f$ for the rank one operator $x \to (x,e)f$, whose norm is $\|e\|\|f\|$. If $\mathcal{N}$ is a nest, then by [8, Lemma 3.7], $e^* \otimes f \in$ alg$\mathcal{N}$ if and only if there is an $E \in \mathcal{N}$

such that $f \in E$ and $e \in (E_-)^\perp$. If $\mathcal{A}$ is a subalgebra of $L(H)$, then we say that $\mathcal{A}$ is a *triangular* operator algebra, if $\mathcal{A} \cap \mathcal{A}^*$ is a maximal abelian selfadjoint subalgebra of $L(H)$. If $\mathcal{J}$ is maximal triangular, and lat$\mathcal{A}$ is a maximal nest, then we say that $\mathcal{A}$ is *strongly reducible.* A *derivation* δ of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{B}$ is a linear map satisfying $\delta(AB) = A\delta(B) + \delta(A)B$, for any $A, B \in \mathcal{A}$. A derivation δ is called $\mathcal{B}$-inner if there exists $T \in \mathcal{B}$, such that $\delta(A) = AT - TA$. When we say that a derivation $\delta : \mathcal{A} \to \mathcal{B}$ is inner, we mean $\mathcal{B}$-inner.

2. Derivations Let $\mathcal{N}$ be a nest. In the following, we consider the derivation from a subalgebra $\mathcal{A}$ of $L(H)$ containing all rank one operators of alg$\mathcal{N}$ into $L(H)$.

Theorem 2.1. *If $\mathcal{N}$ is a nest such that $H_- \neq H$, $\mathcal{A}$ is a subalgebra of $L(H)$ containing $(\text{alg}\mathcal{N}) \cap F_1(H)$, and δ is a derivation from $\mathcal{A}$ into $L(H)$, then δ is inner.*

Proof. Since $H_- \neq H$, for any $f^* \in (H_-)^\perp$, $f^* \neq 0$, we choose y in H such that $f^*(y) = 1$. For any x in H, by [8, Lemma 3.7], it follows that $f^* \otimes x \in \text{alg}\mathcal{N}$. Now define

$$Tx = -\delta(f^* \otimes x)y, \quad \text{for } x \text{ in } H. \tag{2.1}$$

Now for A in $\mathcal{A}$,

$$TAx = -\delta(f^* \otimes Ax)y = -\delta(A)x - A\delta(f^* \otimes x)y = -\delta(A)x + ATx. \tag{2.2}$$

Hence for any $x \in H$, $-TAx + ATx = \delta(A)x$; thus

$$\delta(A) = AT - TA. \tag{2.3}$$

It remains to show that δ is bounded.

Let $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} Tx_n = y$. Now for any rank one operator $A \in \text{alg}\mathcal{N}$, we have that $\delta(A)$ and TA are bounded. It follows that $AT = \delta(A) + TA$ is bounded, and $\lim_{n\to\infty} ATx_n = ATx = Ay$. Since $\mathcal{A}$ contains all rank one operators of alg$\mathcal{N}$, and by [4, Proposition 3.8], every finite rank operator of alg$\mathcal{N}$ is a sum of some rank one operators of alg$\mathcal{N}$, we have, for any finite rank operator B in alg$\mathcal{N}$, $BTx = By$. By [4, Theorem 3.11], choose a bounded net $\{B_\lambda\}$ of finite rank operators in alg$\mathcal{N}$ such that $\lim_\lambda B_\lambda = I$ in the strong operator topology. We have $Tx = y$. By the closed graph theorem, it follows that T is bounded. □

Corollary 2.2. *If $\mathcal{N}$ is a nest such that $0_+ \neq 0$, and $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of* alg$\mathcal{N}$*, then every derivation δ from $\mathcal{A}$ into $L(H)$ is inner.*

Proof. Let $\mathcal{N}^\perp = \{N^\perp : N \in \mathcal{N}\}$. Then $\mathcal{N}^\perp$ is a nest such that $H_- \neq H$. Since $\text{alg}\mathcal{N}^\perp = (\text{alg}\mathcal{N})^*$, it follows that $\mathcal{A}^*$ contains all rank one operators of alg$\mathcal{N}^\perp$. Define $\delta^*(A = (\delta(A^*)))^*$ for any A in $\mathcal{A}^*$.

It is easy to prove that δ^* is a derivation from $\mathcal{A}^*$ into $L(H)$. By Theorem 2.1, we have that δ^* is inner. It follows that δ is inner. □

We now consider a nest $\mathcal{N}$ such that $H_- = H$.

Lemma 2.3. *Let $\mathcal{N}$ be a nest, $E_1, E_2 \in \mathcal{N}$ and $E_1 \subsetneq E_2$. If T is a linear map from E_2 into H such that $ST = TS$ on E_2 for any rank one operator S of* $\operatorname{alg}\mathcal{N}$*, then there exists a λ such that $Tx = \lambda x$, for any $x \in E_1$.*

Proof. For $x \in E_1$, choose $y \in E_2 - E_1$ such that $\|y\| = 1$. Since $y^* \otimes x \in \operatorname{alg}\mathcal{N}$, by hypothesis

$$Ty^* \otimes x(y) = y^* \otimes xTy = Tx = (Ty, y)x. \tag{2.4}$$

Since every one-dimensional subspace of $L(E_2, H)$ is reflexive, it follows that there exists λ such that $T = \lambda I$. □

Lemma 2.4. *Let $\mathcal{N}$ be a nest such that $H_- = H$, and let $M = \cup\{N : N \subsetneq H,\ N \in \mathcal{N}\}$. Then there exists a linear map T from M into H such that $\delta(A)x = (AT - TA)x$, for any x in M.*

Proof. Since $H_- = H$, we may choose an increasing sequence $\{P_i\} \subseteq \mathcal{N}$ such that $P_i \to I$ in the strong operator topology. Also choose $f^* \in P_i^{\perp}$, and $y \in H$, such that $\|f^*\| = 1$, $f^*(y) = 1$, and $\|y\| \le 2$. Define,

$$T_i x = -\delta(f^* \otimes x)y \quad \text{for } x \in P_i. \tag{2.5}$$

Using an argument similar to the proof of Theorem 2.1, we may prove that for A in $\mathcal{A}$, $\delta(A)x = (AT_i - T_iA)x$ for x in P_i. If $j \ge i$, then for $x \in P_i$, $(AT_i - T_iA)x = (AT_j - T_jA)x$. Hence

$$A(T_i - T_j)x = (T_i - T_j)Ax, \quad \text{for } x \in P_i. \tag{2.6}$$

By Lemma 2.3, we have $T_j - T_i = \lambda_{ij}$ on P_{i-1}. Now for $j > 2$, let $\tilde{T}_j = T_1 + \lambda_{1,j}$. We have, for $k > j > 2$, $\tilde{T}_j x = \tilde{T}_k x$ for all $x \in P_{j-1}$. Now for any $x \in \cup\{P_i\} = \cup\{N : N \subsetneq H,\ N \in \mathcal{N}\}$, choose a $j > 2$ such that $x \in P_j$ and let $Tx = \tilde{T}_j x$. Then, T is well defined and for x in M, $\delta(A)x = (AT - TA)x$. □

Remark 2.5. Using the idea in the proof of Theorem 2.1, we can prove that in Lemma 2.3, T_i is a bounded operator from P_i into H.

Theorem 2.6. *If $\mathcal{N}$ is a nest, $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of* $\operatorname{alg}\mathcal{N}$*, and δ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then δ is inner.*

Proof. If $\mathcal{N}$ satisfies $H_- \ne H$, then by Theorem 2.1, we get that δ is inner. If $\mathcal{N}$ satisfies $H_- = H$, then by Lemma 2.4, there exists a linear map T such that

$$\delta(A)x = (AT - TA)x \quad \text{for any } x \text{ in } M = \cup\{N : N \subsetneq H,\ N \in \mathcal{N}\}. \tag{2.7}$$

By (2.5) and the boundedness of δ, it follows that $\|T_i x\| \le 2\|\delta\|\|x\|$. Since $|\lambda_{ij}| \le \|T_i\| + \|T_j\| \le 4\|\delta\|$, it follows that $\|T\| \le 6\|\delta\|$. Thus T is bounded on M. Let $\tilde{T}$ be the unique bounded extension of T to H. Then $\tilde{T}$ is bounded and for A in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. □

Theorem 2.7. *Let $\mathcal{N}$ be a nest satisfying $H_- = H$. If there exists a nontrivial projection $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, and δ is a derivation from $\mathcal{A}$ into $L(H)$, then δ is inner.*

Proof. As in the proof of Lemma 2.4, we choose $P_1 = P$. Let $H = P \oplus P^\perp$. Then T can be decomposed as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{2.8}$$

Let $Q = \cup\{N - P : P \subsetneq N \in \mathcal{N},\ N \neq H\},\ T_{12} : Q \to P,\ T_{22} : Q \to Q$.

By the definition of T, T_{11} and T_{21} are bounded. We now prove that T_{12} and T_{22} are bounded. Since $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathcal{A}$, we have that $\delta(A) = \left(\begin{smallmatrix} 0 & T_{12} \\ -T_{21} & 0 \end{smallmatrix}\right)$ holds on M. Since $\delta(A)$ is bounded, it follows that T_{12} is bounded. Now, for any rank one operator $A \in L(H)$, we have $PA(1-P) \in \mathcal{A}$. Hence,

$$\delta(PA(1-P)) = \begin{pmatrix} PA(1-P) & PA(1-P)T_{22} - T_{11} \\ 0 & -T_{21}PA(1-P) \end{pmatrix} \tag{2.9}$$

holds on M. Since $\delta(PA(1-P))$ is bounded, it follows that $PA(1-P)T_{22}$ is bounded. Hence for any $f^* \in P^\perp$ and $e \in P$, $e \neq 0$, $f^* \otimes eT_{22}$ is bounded on Q. Thus there exists c such that $|f^*(T_{22}x)| \le c$, for any $x \in Q$, and $\|x\| \le 1$. By the uniform boundedness theorem, we have that $\{\|T_{22}x\| : \|x\| \le 1\}$ is bounded. Hence T_{22} is bounded. As in Theorem 2.6, there exists a bounded extension $\tilde{T}$ of T to H such that for A in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. □

3. Applications. In this section, we apply the results above to some special subalgebras of $L(H)$. If $A \supseteq F(H)$, then by Theorem 2.1, we have the following corollaries.

Corollary 3.1 [1]. *Every derivation from a standard operator algebra into $L(H)$ is inner.*

Corollary 3.2 [2]. *If δ is a derivation from* alg$\mathcal{N}$ *into itself, then δ is inner.*

Proof. By Theorems 2.1 and 2.7, we have that there is T in $L(H)$ such that for any A in $\mathcal{A}$, $\delta(A) = AT - TA$. Now we prove that T is in alg$\mathcal{N}$. Now for any P in $\mathcal{N}$, since $\delta(P) = PT - TP$ in alg$\mathcal{N}$, we have that $(I-P)\delta(P)P = 0 = -(I-P)TP$. This completes the proof. □

Let $\mathcal{B}$ be a subalgebra of $L(H)$, and let $\mathcal{S}$ be any subset of $L(H)$. We denote by $C(\mathcal{B},\mathcal{S})$ the collection, $\{T \in L(H) : AT - TA \in \mathcal{S},\ \forall A \in \mathcal{B}\}$.

Lemma 3.3 [6]. *Let $\mathcal{B}$ be a nest algebra and $\mathcal{J}$ be an ideal in $L(H)$. Then $C(\mathcal{B},\mathcal{J}) = CI + \mathcal{J}$.*

Using this lemma and Theorem 2.7, we easily prove the following result.

Corollary 3.4. *If $\mathcal{B}$ is an algebra containing* alg$\mathcal{N}$*, then any derivation $\delta : \mathcal{B} \to C_p$ is inner for $1 \le p \le \infty$.*

Corollary 3.5. *If $\mathcal{B}$ is a triangular operator algebra containing every rank one operator in* alg$\mathcal{N}$*, then every derivation δ from $\mathcal{B}$ into $L(H)$ is inner.*

Proof. Suppose $\tilde{\mathcal{N}}$ is a maximal nest containing $\mathcal{N}$. By hypothesis we have that $B \supseteq (\operatorname{alg}\mathcal{N}) \cap F_1(H) \supseteq (\operatorname{alg}\tilde{\mathcal{N}}) \cap F_1(H)$. Since $\mathcal{B}$ contains all rank one operators of $\operatorname{alg}\mathcal{N}$, we have that $\operatorname{lat}\mathcal{B} \subseteq \mathcal{N}$. By [5, Theorem 4], it follows that $\operatorname{lat}\mathcal{B} = \tilde{\mathcal{N}} = \mathcal{N}$. Since $\mathcal{B}$ is a triangular operator algebra, it follows $\tilde{\mathcal{N}} \subseteq \mathcal{B}$.

If $H_- \neq H$, then by Theorem 2.1, we have that δ is inner.

If $H_- = H$, $\mathcal{N} \subseteq \mathcal{B}$, and $\mathcal{N}$ is a maximal nest, by Theorem 2.7, it follows that δ is inner. □

Remark 3.6. By Corollary 3.1, it follows that every derivation $\delta : F(H) \to L(H)$ is inner. However if $\mathcal{B}$ is a unital algebra containing $F(H)$ and $\mathcal{B} \neq L(H)$, then there is a derivation from $F(H)$ into $\mathcal{B}$ that is not inner, e.g., $\delta = \delta_T$ with $T \notin \mathcal{B}$. Also if $\mathcal{A} = K(H) + \mathbb{C}I$, and $T \notin \mathcal{A}$, then $\delta_T : \mathcal{A} \to \mathcal{A}$ is a derivation that is not inner, but $\mathcal{A}$ contains all rank one operators of $L(H)$.

By [8, Lemma 5.2], we know that if $\mathcal{B}$ is a strongly reducible maximal triangular algebra, then $\operatorname{lat}\mathcal{B}$ is a nest and $\mathcal{B}$ contains all rank one operators of $\operatorname{alglat}(\mathcal{B})$. Hence by Corollary 3.5 and Theorem 2.7, we have the following result.

Corollary 3.7. *Every derivation from a strongly reducible maximal triangular algebra into $L(H)$ is inner.*

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