

HOMOCLINIC ORBITS AND LIE ROTATED VECTOR FIELDS

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ABSTRACT. Based on the definition of Lie rotated vector fields in the plane, this paper gives the property of homoclinic orbit as parameter is changed and the singular points are fixed on Lie rotated vector fields. It gives the conditions of yielding limit cycles as well.

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1. Introduction. The rotated vector fields have been considered as a very important tool which is efficient in the study of the numbers of limit cycles and the distribution of homoclinic orbits. In this paper, we continue with the investigation of the Lie rotated vector fields [6]. Based on the definition of Lie rotated vector fields [6], we give the property of homoclinic orbit as parameter is changed and the singular points are fixed on Lie rotated vector fields. As application of Lie rotated vector fields, we prove that the definition of Lie rotated vector has nothing to do with the change of coordinate. Afterwards, we require the singular points of $X(\mu)$ not to be moved as parameter μ is changed. We study the motion of homoclinic orbits that pass through saddle points on Lie rotated vector fields and the change of a family of periodic orbits that are in the inner neighborhood of homoclinic orbit. Of course, we give some examples to illustrate the concept and notion of Lie rotated vector fields.

2. Topological properties of Lie rotated vector fields. Let $X(\mu)$ be a Lie rotated vector field. In this section, we require the singular points of $X(\mu)$ not to be moved as parameter μ is changed, i.e., the singular points are kept immovable.

THEOREM 2.1. *The Lie rotated vector field is a Lie rotated vector field under the C^2 differential topological transform.*

PROOF. Let ϕ be a C^2 differential topological transform on \mathbb{R}^2 , and let that $\phi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$, $\phi(x) : x \mapsto y = \phi(x)$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let $X(\mu)$ be a Lie rotated vector field, then Y is a corresponding vector field which satisfies formula

$$L(0) \stackrel{\text{def}}{=} X(0) \wedge \{X'_\mu(0) + [X(0), Y]\} > 0 \text{ } (< 0), \quad (2.1)$$

where $X'_\mu(0)$ is the derivative of the vector field $X(\mu)$ at $\mu = 0$. Under the transform ϕ , $X(\mu) = X(x, \mu)$ becomes

$$\tilde{X}(\mu) \stackrel{\text{def}}{=} (\phi_* X)(y, \mu) = D\phi(\phi^{-1}(y)) \cdot X(\phi^{-1}(y), \mu) \quad (2.2)$$

use the same method, under the transform ϕ , Y becomes

$$\bar{Y} \stackrel{\text{def}}{=} (\phi_* Y)(y) = D\phi(\phi^{-1}(y)) \cdot X(\phi^{-1}(y)). \tag{2.3}$$

For $\bar{X}(\mu)$ and the corresponding \bar{Y} , and $\forall y = \phi(x) \in \mathbb{R}^2$, we only need to prove the condition to set up

$$\overline{L(0)} = \bar{X}(0) \wedge \{\bar{X}'_\mu(0) + \overline{[\bar{X}(0), \bar{Y}]}\} > 0 \text{ (} < 0 \text{)}, \tag{2.4}$$

where $\overline{[\ , \]}$ is Lie bracket, which expresses the action to the variable y . Since

$$\bar{X}(0) \wedge \overline{[\bar{X}(0), \bar{Y}]} = D\phi(x) \cdot X(x, 0) \wedge [X(x, 0), Y(x)], \tag{2.5}$$

$$\bar{X}(0) \wedge \bar{X}'_\mu(0) = D\phi(x) \cdot X(x, 0) \wedge X'_\mu(x, 0), \tag{2.6}$$

where $x = \phi^{-1}(y)$. From formulae (2.5) and (2.6), we can find that

$$\bar{L}(0) = D\phi(x) \cdot L(0), \tag{2.7}$$

where $x = \phi^{-1}(y)$. But ϕ is the C^2 differential topological transform, $D\phi(x) \neq 0$, and $L(0) > 0$ (or < 0), it follows that $\bar{L}(0) > 0$ (or < 0). \square

Theorem 2.1 illustrates that Lie rotated vector fields have nothing to do with the choice of coordinates, and the symbol of $\bar{L}(0)$ in formula (2.4) is decided by the symbol of $L(0)$ and the formula of Jacobi on the transform ϕ .

THEOREM 2.2. *Let $X(\mu)$ be a Lie rotated vector field, $X(0)|_{p_0} = 0$.*

- (1) *If the index $J_0(p_0)$ of singular point p_0 of $X(0)$ is not equals to zero, then $Y|_{p_0} = 0$.*
- (2) *If the index $J_0(p_0)$ of singular point p_0 of $X(0)$ is changed as μ is changed, then $Y|_{p_0} \neq 0$.*

PROOF. (1) Using reduction to absurdity, we let $Y|_{p_0} \neq 0$. From [6, Lemma 2.1], we know that $\psi_*^\mu X(\mu)|_{\psi^\mu(p_0)} = 0$ ($\mu \neq 0$), the proof is similar to [6, part (2) of Theorem 3.2]. For $\forall \varepsilon > 0$, such that $\psi^\mu(p_0) \notin \overline{S_\varepsilon(p_0)}$, $|\mu| < \bar{\delta} < \delta$, $\bar{\delta}$ is given by [6, Theorem 3.2]. Since $\psi_*^\mu X(\mu)|_{p_0} \neq 0$, it is noted that $J_{\mu_*}(p_0) = 0$ ($\mu \neq 0$) to $\partial S_\varepsilon(p_0)$ about $\psi_*^\mu X(\mu)$, but $J_0(p_0) \neq 0$ to $\partial S_\varepsilon(p_0)$ about $X(0)$. From [6, Lemma 2.3], we know that this is a contradiction, thus $Y|_{p_0} = 0$.

(2) The proof is similar to (1). In fact, we might as well let $Y|_{p_0} = 0$, then $\psi_*^\mu X(\mu)|_{p_0} = 0$. Let $J_0(p_0) = r_0$ about $X(0)$, but $J_\mu(p_0) = r_\mu$ ($r_\mu \neq r_0$, $\mu \neq 0$) about $X(\mu)$. From [6, Lemma 2.2], we know that $J_{*\mu}(p_0) = r_\mu$. Again from [6, Lemma 2.3], when $|\mu| < \bar{\delta} < \delta$, $J_0(p_0) = J_{*\mu}(p_0) = r_0$, i.e., $r_0 = r_\mu$, and by the supposition $r_0 \neq r_\mu$, this is a contradiction, thus $Y|_{p_0} \neq 0$. \square

COROLLARY 2.3. *Let $X(\mu)$ be a Lie rotated vector field, if $p_{01}, p_{02}, \dots, p_{0m}$ are m elementary singular points of $X(0)$, then the corresponding vector field Y , certainly set up*

$$Y|_{p_{0j}} = 0 \quad (1 \leq j \leq m). \tag{2.8}$$

3. The motion of homoclinic orbit. From the theory of structural stability on two-dimensional manifolds, we know that the systems which have homoclinic orbit passing through the saddle points are structurally unstable, this orbit which links saddle points can extremely be burst under trouble, thus can change the topological structure of the orbit. When $X(\mu)$ is Lie rotated vector field, and if $\mu = 0$, $X(0)$ has homoclinic orbit passing through saddle points, we consider the change of topological structure of the orbit of $X(\mu)$ when $\mu \neq 0$.

THEOREM 3.1. *Let $X(\mu)$ be Lie rotated vector field, $\Gamma_0 = \{q(t), t \in \mathbb{R}\} \cup p_0$ is homoclinic orbit passing through the hyperbolic saddle point p_0 on limited region. If $\sigma_0 = \text{div} X(0)|_{p_0} \neq 0$, then when μ is changed toward suitable direction, Γ_0 disappears, but it will produce unique limit cycle of $X(\mu)$ in the neighborhood of Γ_0 , and the limit cycle is stable or unstable; but when μ is changed towards other direction, Γ_0 disappears, and it will not produce any limit cycle of $X(\mu)$ in the neighborhood of Γ_0 .*

PROOF. We might as well let Γ_0 positively oriented, $\sigma_0 < 0$. From [5, Theorem 1], we know that Γ_0 is inner stable. Since p_0 is a hyperbolic saddle point of $X(0)$, from Corollary 2.3, we know that it certainly has $Y|_{p_0} = 0$, thus it has $\psi_*^\mu X(\mu)|_{p_0} = 0$ ($\mu \neq 0$). $\psi_*^\mu X(\mu)$ has homoclinic orbit Γ_0 passing through a hyperbolic saddle point p_0 when $\mu = 0$. By [6, Lemma 2.3], when $0 < |\mu| < \delta$, Γ_0 is burst in $\psi_*^\mu X(\mu)$ ($\mu \neq 0$), yet from the topological equivalence of orbital structure of $X(\mu)$ and orbital structure of $\psi_*^\mu X(\mu)$, we know that Γ_0 of $X(\mu)$ is burst when $\mu \neq 0$, i.e., when $0 < \mu < \delta$, the homoclinic orbit Γ_0 passing through hyperbolic saddle point p_0 is burst into stable manifold S_μ and unstable manifold U_μ . Since Γ_0 is inner stable, by continuous dependence of solution to the parameter μ and using the Poincaré-Bendixson annular regional theorem, we prove that there is a limit cycle in the neighborhood of Γ_0 when $\mu > 0$, yet by $\sigma_0 < 0$, we know that there is only a unique stable limit cycle in the neighborhood of Γ_0 . But when $\mu < 0$, $|\mu| < \delta$, let there be a limit cycle in the neighborhood of Γ_0 , this is the same discussion as above, we know that it is sure to have semi-stable limit cycle or unstable limit cycle, this is contradiction with $\sigma_0 < 0$. □

Using Theorems 2.2 and 3.1, we can easily prove the following corollary.

COROLLARY 3.2. *Let $X(\mu)$ be a Lie rotated vector field, $\Gamma_0 = \{q(t), t \in \mathbb{R}\} \cup p_0$ is homoclinic orbit passing through the saddle point p_0 of $X(0)$ on the limited region, $\sigma_0 = \text{div} X(0)|_{p_0} \neq 0$. If $J_0(p_0) \neq 0$ (or $J_0(p_0) = 0$, but $Y|_{p_0} = 0$), then when μ is changed towards a suitable direction, Γ_0 disappears, but it will produce a unique limit cycle of $X(\mu)$ in the neighborhood of Γ_0 , and the limit cycle is stable or unstable; but when μ is changed towards the other direction, Γ_0 disappears, and it will not produce any limit cycle of $X(\mu)$ in the neighborhood of Γ_0 .*

Using [1, Theorems 45 and 49], [4, Theorem 1.2], and Theorem 2.2 of this paper, we have the following corollary.

COROLLARY 3.3. *Let $X(\mu)$ be a Lie rotated vector field, $\Gamma_0 = \{q(t), y \in \mathbb{R}\} \cup p_0$ is homoclinic orbit passing through saddle point p_0 of $X(0)$ on the limited region, $\sigma_\mu = \text{div} X(\mu)|_{p_0} = 0$ ($|\mu| \ll 1$ and $|\mu| < \delta$), $I_0 = \int_{\Gamma_0} \sigma_0 dt \neq 0$. If $J_0(p_0) \neq 0$ (or $J_0(p_0) = 0$,*

but $Y|_{p_0} = 0$), then when μ is changed towards a suitable direction, Γ_0 disappears, but it will produce a unique limit cycle of $X(\mu)$ in the neighborhood of Γ_0 , and the limit cycle is stable or unstable; but when μ is changed towards other direction, Γ_0 disappears, and it will not produce any limit cycle of $X(\mu)$ in the neighborhood of Γ_0 .

EXAMPLE 3.4. Let $X(\mu) = (2x_2, 2x_1 - 3x_1^3 - x_2(x_1^3 - x_1^2 + x_2^2) + \mu x_2^3)$, when $\mu = 0$, $p_0 = (0, 0)$, $p_1 = (2/3, 0)$ are elementary singular points of $X(0)$, where p_0 is saddle point, p_1 is an unstable focal point. Since $\Gamma_0 : x_1^3 - x_1^2 + x_2^2 = 0$ is homoclinic orbit passing through the hyperbolic saddle point p_0 of $X(0)$ (this example is shown in [5]). By Corollary 2.3, we can take $Y = (0, \beta(2x_1 - 3x_1^2))$, where $\beta \in \mathbb{R}$. For, $\forall \varepsilon > 0$, ε is taken small enough, we make open neighborhoods $S_\varepsilon(p_0)$ and $S_\varepsilon(p_1)$ of p_0 and p_1 , respectively, then there is a limited region $D \subset \mathbb{R}^2$, $\Gamma_0 \subset D$, order $\beta = \varepsilon^3 > 0$, at the ordinary point of $X(0)$ of $D \setminus \{S_\varepsilon(p_0) \cup S_\varepsilon(p_1)\}$, we have

$$L(0) = 2\beta(2x_1 - 3x_1^2 + x_2^3)^2 + 2\beta(4x_2^2(1 - 3x_1) - x_2^6) + 2x_2^4 > 0, \quad (3.1)$$

i.e., $X(\mu)$ constitutes Lie rotated vector field on D . Take $|\mu| \ll 1$ and $|\mu| < \delta$, note $\sigma_\mu = \text{div} X(\mu)|_{p_0} = 0$, again from [5], we can know that $I_0 = \int_{\Gamma_0} \sigma_\mu dt < 0$. By Corollary 3.3, when $\mu < 0$, Γ_0 disappears, but it will produce a unique stable limit cycle of $X(\mu)$ in the neighborhood of Γ_0 ; but when $\mu > 0$, Γ_0 disappears, and it will not produce any limit cycle in the neighborhood of Γ_0 .

Theorem 3.1 and Corollary 3.2 require $\sigma_0 \neq 0$, Corollary 3.3 requires $\sigma_\mu = 0$ ($0 \leq |\mu| < \delta$ and $|\mu| \ll 1$), using [3, Lemmas 8 and 9] and the proven method of Theorem 3.1 in this paper, we have the following corollary.

COROLLARY 3.5. Let $X(\mu)$ be a Lie rotated vector field, $\Gamma_0 = \{q(t), t \in \mathbb{R}\} \cup p_0$ is homoclinic orbit passing through the saddle point p_0 of $X(0)$ on the limited region, Γ_0 is stable (unstable). If $J_0(p_0) \neq 0$ (or $J_0(p_0) = 0$, but $Y|_{p_0} = 0$), then when μ is changed towards a suitable direction (towards the other direction), Γ_0 disappears, but it at least produces a limit cycle in the neighborhood of Γ_0 , the limit cycle is stable (unstable); when μ is changed towards the other direction (towards a suitable direction), Γ_0 disappears, but it will not produce any limit cycle in the neighborhood of Γ_0 .

EXAMPLE 3.6. Let $X(\mu) = (2x_2, 2x_1 - 3x_1^2 - x_2(x_1^3 - x_1^2 + x_2^2) + \mu x_2)$, when $\mu = 0$, the state of $X(0)$ is same as Example 3.4, we yet take $Y = (0, \beta(2x_1 - 3x_1^2))$. For, $\forall \varepsilon > 0$, ε is sufficiently small, order $\beta = \varepsilon > 0$, then there is a region $D \subset \mathbb{R}^2$, $\Gamma_0 \subset D$, at the ordinary point of $X(0)$, we have

$$L(0) = 2\beta(2x_1 - 3x_1^2 + x_2^3)^2 + 2\beta(4x_2^2(1 - 3x_1) - x_2^6) + 2x_2^2 > 0, \quad (3.2)$$

i.e., $X(\mu)$ constitutes a Lie rotated vector field on D . Take $|\mu| \ll 1$, $|\mu| < \delta$, by $\sigma_0 = \text{div} X(0)|_{p_0} = 0$, $\sigma_\mu = \text{div} X(\mu)|_{p_0} = \mu$ ($\mu \neq 0$), and $I_0 = \int_{\Gamma_0} \sigma_\mu dt < 0$, we know that Γ_0 is inner stable, $J_0(p_0) = -1 \neq 0$. From Corollary 3.5, when $\mu < 0$, in the neighborhood of Γ_0 , at least produces a limit cycle; when $\mu > 0$, in the neighborhood of Γ_0 , it does not produce any limit cycle.

Now we consider that $X(\mu)$ is a Lie rotated vector field. If $\mu = 0$, Γ_0 is homoclinic orbit passing through saddle point p_0 of $X(0)$ on the limited region, and the inner neighborhood of Γ_0 is imbued a family of periodic orbits, where $\sigma_0 = I_0 = 0$.

LEMMA 3.7. *Let X be a C^1 vector field, the limited region $\Delta_1 \subset \mathbb{R}^2$ is imbued a family of periodic orbits L_h for X , $h \in (a, b) \subset \mathbb{R}$, then for all C^1 vector fields Y , we have*

$$A_0(h) = \int_{L_h} (X \wedge [X, Y]) \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} dt = 0, \tag{3.3}$$

where $h \in (a, b)$.

PROOF. Using the formula of Theorem 2.3 of Chapter 3 in §2 of [2], we have

$$X \wedge [X, Y] = \langle X, \nabla(X \wedge Y) \rangle - (X \wedge Y) \cdot \operatorname{div} X. \tag{3.4}$$

Both sides of (3.4) are multiplied by the factor $\exp \left\{ - \int_0^t \operatorname{div} X dt \right\}$, and both sides are integrated along the circuit of L_h , let the period of L_h be $\omega(h)$, using [7, formula (2.16)], we have

$$\begin{aligned} & \int_{L_h} (X \wedge [X, Y]) \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} dt \\ &= \int_{L_h} \langle X, \nabla(X, Y) \rangle \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} dt + \int_{L_h} (X \wedge Y) \cdot \frac{d}{dt} \left\{ \exp \left(- \int_0^t \operatorname{div} X dt \right) \right\} dt \\ &= \int_{L_h} \langle X, \nabla(X, Y) \rangle \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} dt - \int_{L_h} \frac{d}{dt} (X \wedge Y) \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} dt \\ & \quad + \left[(X \wedge Y) \cdot \exp \left\{ - \int_0^t \operatorname{div} X dt \right\} \right]_0^{\omega(h)} = 0. \end{aligned} \tag{3.5}$$

This proof is completed. □

THEOREM 3.8. *Let $X(\mu)$ be a Lie rotated vector field, $\Gamma_0 = \{q(t), t \in \mathbb{R}\} \cup p_0$ is a homoclinic orbit passing through the saddle point p_0 of $X(0)$ on limited region. If the inner neighborhood $\Delta \subset \mathbb{R}^2$ of Γ_0 is imbued a family of periodic orbits $L_h, h \in (a, b) \subset \mathbb{R}$, then when $\mu \neq 0$, the inner neighborhood Δ of Γ_0 will not produce any closed orbit of $X(\mu)$.*

PROOF. By [7, Theorem 4.10], when $\mu \neq 0$, on the undisappeared L_{h_0} , we have

$$A_1(h_0) = \int_{L_{h_0}} (X(0) \wedge X'_\mu(0)) \cdot \exp \left\{ - \int_0^t \operatorname{div} X(0) dt \right\} dt = 0, \tag{3.6}$$

where $h_0 \in (a, b)$. But from Lemma 3.7, we have

$$A_1(h_0) = A_1(h_0) + A_0(h_0) = \int_{L_{h_0}} L(0) \cdot \exp \left\{ - \int_0^t \operatorname{div} X(0) dt \right\} dt \neq 0. \tag{3.7}$$

It follows that the proof is completed. □

EXAMPLE 3.9. Let $X(\mu) = (2x_2, 2x_1 - 3x_1^2 + \mu x_2)$, when $\mu = 0$, $p_0 = (0, 0)$ is hyperbolic saddle point of $X(0)$, $p_1 = (2/3, 0)$ is the center of $X(0)$, $\Gamma_0 : x_1^3 - x_1^2 + x_2^2 = 0$ is homoclinic orbit passing through p_0 of $X(0)$, yet take $Y = (0, \kappa(2x_1 - 3x_1^2))$. When κ

is taken arbitrary sufficient small, $0 < \kappa \ll 1$, then there is region $D \subset \mathbb{R}^2$, $\Gamma_0 \subset D$, at all ordinary points of $X(0)$ on D , we have

$$L(0) = 2x_2^2(1 + 4\kappa(1 - 3x_1)) + 2\kappa(2x_1 - 3x_1^2)^2 > 0, \quad (3.8)$$

i.e., $X(\mu)$ constitutes a Lie rotated vector field on D . From Theorem 3.8, we can know that there is not closed orbit in the inner neighborhood of Γ_0 when $\mu \neq 0$.

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