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# A REPRESENTATION THEOREM FOR THE LINEAR QUASI-DIFFERENTIAL EQUATION $(py')' + qy = 0$

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Abstract. We establish a representation for $q$ in the second-order linear quasi-differential equation $(py')' + qy = 0$. We give a number of applications, including a simple proof of Sturm's comparison theorem.

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**1. Introduction.** We are concerned with the quasi-differential equation

$$(p(x)y')' + q(x)y = 0 \tag{1.1}$$

over a half-line $[a,\infty)$, where $1/p$, $q$ are locally Lebesgue integrable over $(a,\infty)$ and $p$ is a positive function.

By the term solution we mean a nontrivial real valued function $\phi$ that satisfies (1.1) almost everywhere in $(a,\infty)$ and $\phi$ and $p\phi'$ are locally absolutely continuous over $[a,\infty)$. For a discussion of existence and uniqueness properties of the solutions of (1.1), see Naimark [3].

It is well known that there is a strong relationship between the oscillatory behaviour of the solutions of (1.1) and the existence of solutions to the corresponding Riccati equation. A great deal has been written about this connection, see, Reid [7, Chapter 4] or Willet [8].

We find the following notation convenient: let $\Omega_a$ denote the space of functions positive and $AC_{\text{loc}}[a,\infty)$ such that if $\omega \in \Omega_a$, then $p\omega' \in AC_{\text{loc}}[a,\infty)$. If $\omega \in \Omega_a$, define a function $v$ by $v = -p\omega'/2\omega$.

With the above assumptions, either a solution of (1.1) has infinitely many zeros, then every solution of (1.1) has this property, or every solution has at most a finite number of zeros. In the former case, we say that (1.1) is oscillatory, writing $(p,q)$ is oscillatory, and in the latter case, we say that (1.1) is nonoscillatory, writing $(p,q)$ is nonoscillatory.

In the following section, we extend the relationship between the Riccati equation and the quasi-differential equation (1.1). We then show how this extension can be used to distinguish between the mutually exclusive oscillatory behaviour of the solutions of (1.1). Finally, we see how this extension may be interpreted as a representation result, giving a simple proof to Sturm's comparison theorem and a result of Hartman, under the less stringent hypothesis of integrably (mentioned above) rather than the classical conditions of continuity. We also show how to construct oscillatory quasi-differential equations of the form (1.1) without explicitly finding the solutions of such equations.

**2. Main results.** We begin by quoting a representation result from O'Hara [4, Lemma 2].

**Proposition 2.1.** *There is a $\omega \in \Omega_a$ and a real number $k \neq 0$ such that*

$$q = -v' - \frac{v^2}{p} + \frac{k^2\omega^2}{p}. \tag{2.1}$$

**Proof.** Let $\phi, \theta$ be linearly independent solutions of (1.1) and let $\omega = (\phi^2 + \theta^2)^{-1}$. A straightforward calculation gives (2.1). □

**Remark 2.2.** The function $\omega$ is not unique and may be chosen so that $\omega(a)$ has any prescribed value.

The proposition fails when $k = 0$ then we have Wintner's result [9], which may also be regarded as a representation result.

**Proposition 2.3.** *There exists $\omega \in \Omega_a$ such that*

$$q = -v' - \frac{v^2}{p} \tag{2.2}$$

*over $[b, \infty)$, $b > a$ if and only if (1.1) has a nontrivial solution with no zero in $[b, \infty]$.*

**Proposition 2.4.** *Let $\omega \in \Omega_a$ and $c \neq 0$ any real number. Then the general solution of*

$$(py')' - \left(v' + \frac{v^2 - c^2\omega^2}{p}\right)y = 0 \tag{2.3}$$

*is given by*

$$y(x) = \frac{A}{\sqrt{\omega(x)}} \sin\left[c\int_a^x \frac{\omega}{p}\,dt + \alpha\right], \tag{2.4}$$

*where $A$ and $\alpha$ are constants.*

**Proof.** Define a function $y$ by

$$y(x) = \exp\left[\int_a^x \frac{v + \lambda\omega}{p}\,dt\right], \tag{2.5}$$

where $\lambda$ is a real or a complex constant. Then

$$py' = (v + \lambda\omega)y. \tag{2.6}$$

Differentiating (2.6) and using $p\omega' = -2v\omega$, we have

$$(py')' = \left(v' + \frac{v^2 + \lambda^2\omega^2}{p}\right)y. \tag{2.7}$$

Taking $\lambda = ic$, we have the following solutions to (2.3):

$$\begin{aligned} \phi(x) &= \exp\left[\int_a^x \frac{v}{p}\,dt\right]\sin\left[c\int_a^x \frac{\omega}{p}\,dt\right],\\ \theta(x) &= \exp\left[\int_a^x \frac{v}{p}\right]\cos\left[c\int_a^x \frac{\omega}{p}\,dt\right]. \end{aligned} \tag{2.8}$$

A straightforward calculation shows that $y = c_1\theta_1 + c_2\theta$ gives (2.4). □

The following fundamental result which is a direct consequence of Propositions 2.1 and 2.4 gives necessary and sufficient conditions for the oscillation and nonoscillation of solution of (1.1).

**Corollary 2.5.** *Suppose $\omega \in \Omega_a$ and $c > 0$ be chosen, so that*

$$q = -v' - \frac{v^2}{p} + \frac{c\omega^2}{p}. \tag{2.9}$$

*Then*

(a) *$(p,q)$ is nonoscillatory if and only if $\int^\infty \omega/p < \infty$;*
(b) *(p, q) is oscillatory if and only if $\int^\infty \omega/p = \infty$.*

**Remark 2.6.** Using techniques similar to those outlined in O'Hara [5] and O'Hara and Payne [6] and the above results, we can distinguish very effectively between oscillation and nonoscillation.

**Remark 2.7.** For the sake of completeness, we briefly consider the case $c = 0$. On this occasion, we take

$$\phi(x) = \int_a^x \frac{v}{p}\,dt \qquad \text{and} \qquad \theta(x) = \phi(x)\int_a^x \frac{v}{p}\,dt. \tag{2.10}$$

A similar calculation to the one used in Proposition 2.4 shows that

$$\frac{1}{\sqrt{\omega(x)}} \qquad \text{and} \qquad \frac{1}{\sqrt{\omega(x)}}\int_a^x \frac{\omega}{p}\,dt \tag{2.11}$$

are independent solutions to

$$(py')' - \left(v' + \frac{v^2}{p}\right)y = 0. \tag{2.12}$$

## 3. Applications.

Consider another quasi-differential equation of the form (1.1)

$$(p_1(x)y')' + q_1(x)y = 0 \tag{3.1}$$

over the same half-line as (1.1), where $p_1$ is a positive function and $1/p_1, q_1 \in L_{\text{loc}}[a,\infty)$. We now give an alternate proof of Sturm's comparison theorem.

**Theorem 3.1.** *Let $p_1 < p$ and $q_1 > q$ on the half-line $[a,\infty)$. If $(p,q)$ is oscillatory, then $(p_1,q_1)$ is oscillatory.*

**Proof.** By Proposition 2.4, we can find $\omega \in \Omega_a$ and a constant $c > 0$ such that equation (2.9) holds, and $\int^\infty \omega/p = \infty$. On the contrary, suppose that $(p_1, q_1)$ is nonoscillatory. Then, by Proposition 2.3, we can find a $u \in \Omega_a$ such that

$$q_1 = -u' - \frac{u^2}{p_1}. \tag{3.2}$$

Define a function $\psi$ by $u = v + \omega\psi$. It follows that

$$u' = v' - \frac{2v\omega\psi}{p} + \omega\psi'. \tag{3.3}$$

Then

$$\begin{aligned} q_1 &= -u' - \frac{u^2}{p} + \left(\frac{1}{p} - \frac{1}{p_1}\right)u^2 \\ &= -v' + \frac{2v\omega\psi}{p} - \omega\psi' - \frac{v^2 + 2v\omega\psi + \omega^2\psi^2}{p} + \left(\frac{1}{p} - \frac{1}{p_1}\right)u^2 \\ &= -v' - \frac{v^2}{p} - \omega\psi' - \frac{\omega^2\psi^2}{p} + \left(\frac{1}{p} - \frac{1}{p_1}\right)u^2. \end{aligned} \tag{3.4}$$

By hypothesis, $q_1 > q$, hence by (2.9) and (3.4) we have

$$-\omega\psi' - \frac{\omega^2\psi^2}{p} + \left(\frac{1}{p} - \frac{1}{p_1}\right)u^2 > \frac{c\omega^2}{p} \tag{3.5}$$

implying

$$\psi' + \frac{\omega\psi^2}{p} + \frac{c\omega}{p} < 0, \tag{3.6}$$

since $p_1 < p$ and $\omega$ is positive. Rearranging inequality (3.6) gives

$$-\frac{\psi'}{\psi^2 + c} > \frac{\omega}{p}. \tag{3.7}$$

Integrating inequality (3.7), we have

$$\left[-\frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{\psi}{\sqrt{c}}\right)\right]_a^x > \int_a^x \frac{\omega}{p}\,dt. \tag{3.8}$$

This leads to a contradiction since the function $\tan^{-1}$ is bounded and yet we know that $\int^\infty \omega/p = \infty$. This completes the proof. □

An immediate consequence of Theorem 3.1 is the following.

**Corollary 3.2.** *If $p_1 > p$ and $q_1 < q$ on the half-line $[a, \infty)$ and $(p, q)$ is nonoscillatory, then $(p_1, q_1)$ is nonoscillatory.*

**Remark 3.3.** It is possible to construct oscillatory quasi-differential equations of the form (1.1), without having to solve the equation.

**Example 3.4.** Consider the differential equation

$$y'' + \left(\frac{1}{4x^2} + \frac{c+(1/4)}{(x\ln x)^2}\right)y = 0 \tag{3.9}$$

over $(1,\infty)$, where $c$ is a positive constant.
Letting $\omega(x) = (x\ln x)^{-1}$. A simple calculation shows that

$$v(x) = \frac{1}{2x} + \frac{1}{2x\ln x}. \tag{3.10}$$

Furthermore,

$$q(x) = -v'(x) - v^2(x) + c\omega^2(x) = \frac{1}{4x^2} + \frac{c+(1/4)}{(x\ln x)^2} \tag{3.11}$$

gives the coefficient of $y$ in equation (3.9). We have

$$\int^{\infty} \omega = \int^{\infty} \frac{dx}{x\ln x} = \infty. \tag{3.12}$$

Hence by Corollary 2.5, equation (3.9) is oscillatory. Also, notice that we are unable to use Leighton oscillation criteria [2] to imply oscillation, since

$$\int^{\infty} q = \int^{\infty} \left(\frac{1}{4x^2} + \frac{c+(1/4)}{(x\ln x)^2}\right) dx < \infty. \tag{3.13}$$

**Remark 3.5.** Finally, the representation results can be used to prove a well-known result due to Hartman [1, page 354], in the case of the linear quasi-differential equation (1.1).

**Proposition 3.6.** *If $\phi,\theta$ form a fundamental set of solutions of equation (1.1), then $(p,q)$ is oscillatory or nonoscillatory as*

$$\int^{\infty} \frac{1}{p(\phi^2+\theta^2)} \tag{3.14}$$

*diverges or converges.*

**Proof.** Take $\omega = (\phi^2+\theta^2)^{-1}$ in Proposition 2.4. □

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